(Re)Painting (and Polarizing) Arrow Types

the road to **Interaction Improvement**

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Denotational Semantics

Interpretation: program t $\rightsquigarrow$ $[\![t]\!]$
Abstracting away the implementation details

Equivalence: programs t and u $\rightsquigarrow$ $[\![t]\!] = [\![u]\!]$
Equating programs without looking at implementation details

Induces an *equational theory*:

- *Compatible*:

$$\text{if } [\![t]\!] = [\![u]\!] \text{ then } \forall C,\ [\![C\langle t\rangle]\!] = [\![C\langle u\rangle]\!];$$

- *Invariant under computation*:

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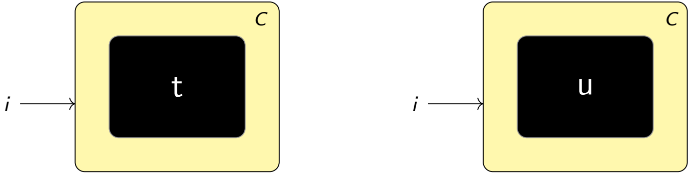
Contextual Equivalence

t u

When are two λ-terms equivalent? [Mor68]

$t \equiv^{\mathrm{ctx}} u$ if for all contexts C . $[\ C\langle t\rangle \Downarrow \;\Leftrightarrow\; C\langle u\rangle \Downarrow\ ]$

Contextual Equivalence



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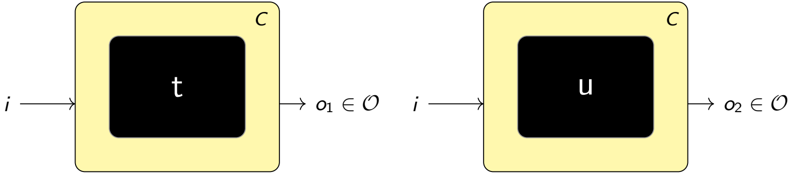

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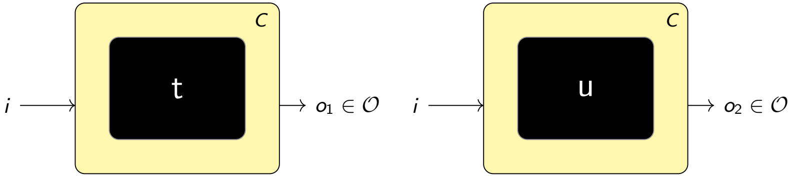

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Denotational Semantics vs. Contextual Equivalence

$$[\![t]\!] = [\![u]\!] \overset{?}{\iff} t \equiv^{\mathrm{ctx}} u$$

- Sometimes it matches!

  Scott's $\mathcal{D}_\infty$ model $\sim$ observing head termination

- Sometimes it does not...

  Game semantics, Relational Model

Can we get a contextual characterization of these models?

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Intensional Information in (Some) Denotational Models

These models may contain *intensional* information:

$$x : \mathtt{int} \cap \mathtt{int} \vdash x + x : \mathtt{int}$$

vs.

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Some programs behave the same but use different resources.
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Going Too Intensional

Sands' **cost equivalence** [San99] is defined as:

$$t \equiv^{\mathrm{cost}} u \quad \text{if } \forall C, \forall k \geq 0\,.\quad [\ C\langle t\rangle \Downarrow_{\mathcal{O}}^{k} \quad \Leftrightarrow \quad C\langle u\rangle \Downarrow_{\mathcal{O}}^{k}\ ]$$

Not an equational theory if we observe termination!
$\mathtt{I} =_{\beta} \mathtt{I}\,\mathtt{I}$ but $\mathtt{I} \not\equiv^{\mathrm{cost}} \mathtt{I}\mathtt{I}$

Where do **relational semantics** fit?

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The best of both worlds

A cost-sensitive equational theory?

How can we measure the interaction between a program and a context modulo the internal dynamics?

Checkers λ-Calculus [POPL 2025]:
a framework to identify
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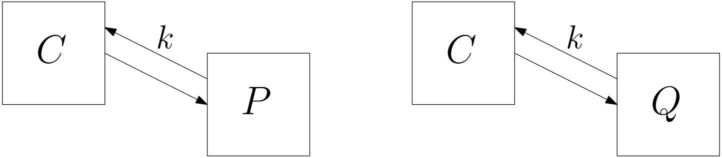

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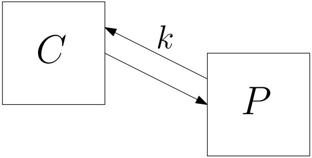

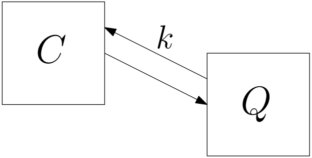

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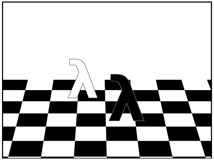

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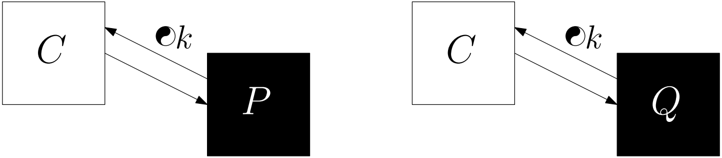

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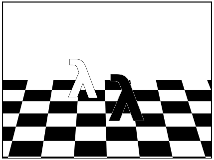

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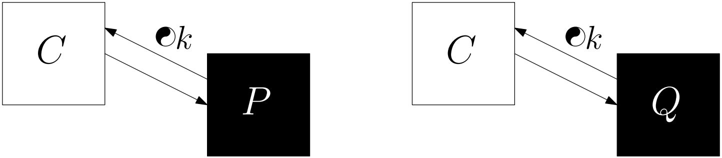
What should be the meaning of a program

Interaction Equivalence is an equational theory!



$$P \equiv^{\text{int}} Q$$

- Duality between Program/Context reminiscent of Game Semantics
- Modeling communication $P \,|\, C$ akin to π-calculus and LTS

Inspecting Black Boxes

Contextual equivalences are hard to check! $\forall$-quantifier

Intensional equivalences are easy to check!

$\rightarrow$ Böhm tree equivalence, normal form bisimilarity, set of approximants, etc.

Tree Characterization [POPL 2025]:

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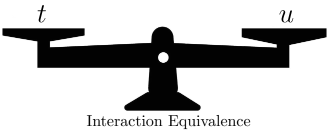

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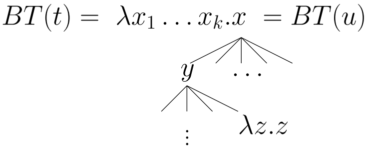
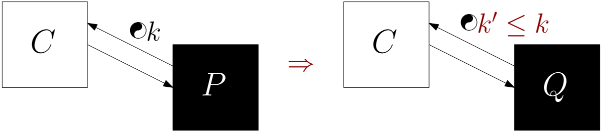


Interaction Improvement

Today[FoSSaCS 2026]!: studying optimizations



$$P \sqsubseteq^{\mathrm{int}\cdot\mathrm{imp}} Q$$

Tree Characterization:

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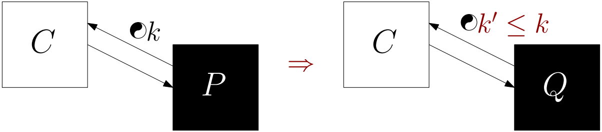

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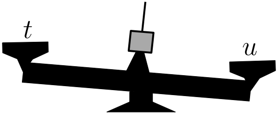

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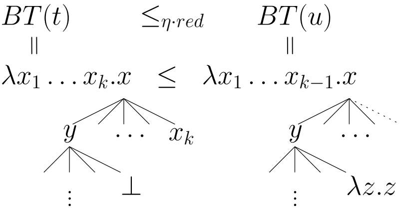

Characterizing the Relational Preorder: $[\![t]\!]_{\text{rel}} \subseteq [\![u]\!]_{\text{rel}}$

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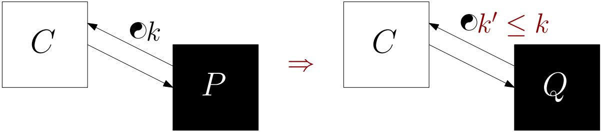

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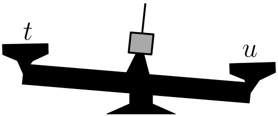

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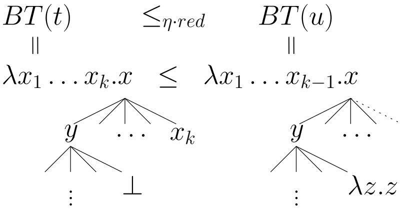

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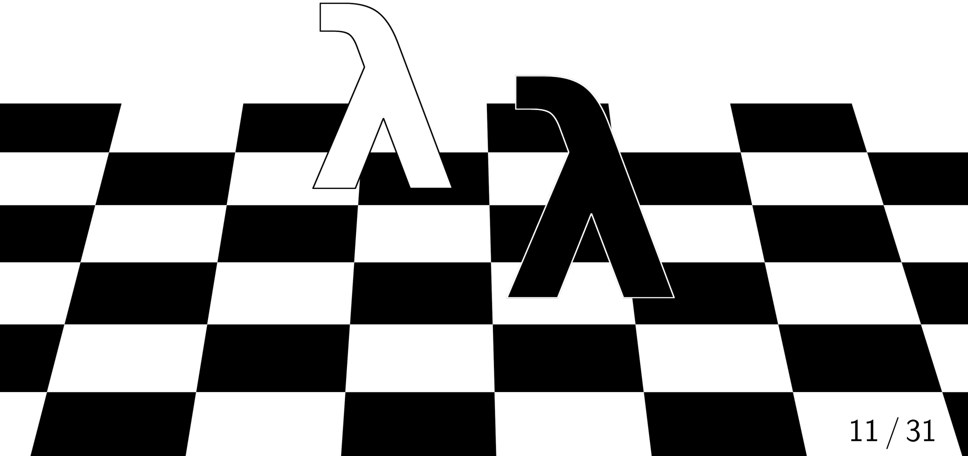

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\Lambda \ni t, u & := & x & \mid \lambda x.t & \mid tu \\
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\end{array}
$$

Silent Steps:

$(\lambda_{\bullet} x.t) \bullet u \mapsto_{\beta\tau} t\{x := u\}$

$(\lambda_{\circ} x.t) \circ u \mapsto_{\beta\tau} t\{x := u\}$

Interaction Steps:

$(\lambda_{\bullet} x.t) \circ u \mapsto_{\beta\LEFTcircle} t\{x := u\}$

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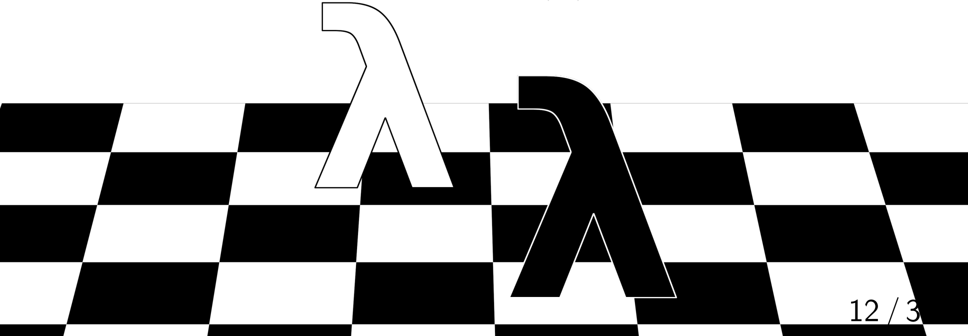
Intuition: $\overline{C}^{\circ}\langle t^{\bullet}\rangle$



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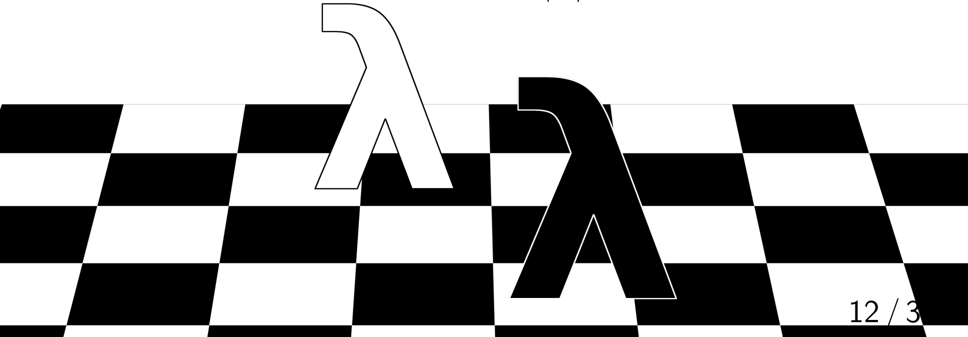

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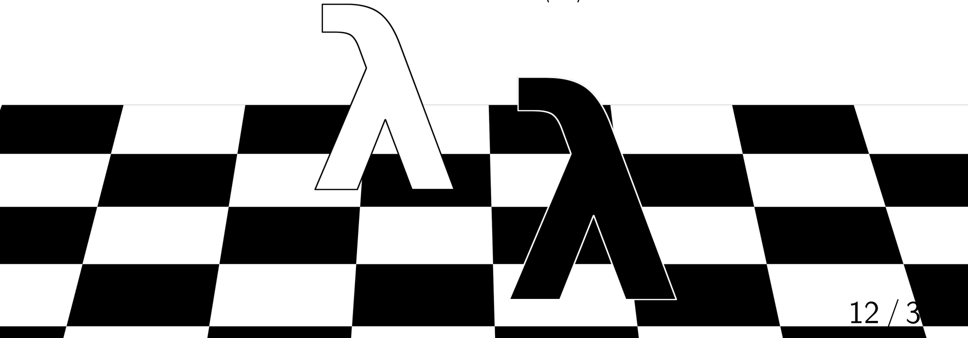

Interaction Cost?

Counting interactions depends on the reduction sequence.

$$\begin{tikzcd}
& (\lambda_{\bullet} x. x \bullet x) \bullet (\mathtt{I}_{\circ} \bullet \mathtt{I}_{\bullet}) \arrow[dl, "\textbf{1 interaction step}"'] \arrow[dr, "\text{1 silent step}"] & \\
(\lambda_{\bullet} x. x \bullet x) \bullet \mathtt{I}_{\bullet} \arrow[dr, "\text{1 silent step}"'] & & (\mathtt{I}_{\circ} \bullet \mathtt{I}_{\bullet}) \bullet (\mathtt{I}_{\circ} \bullet \mathtt{I}_{\bullet}) \arrow[dl, "\textbf{2 interaction step}"] \\
& \mathtt{I}_{\bullet} \bullet \mathtt{I}_{\bullet} &
\end{tikzcd}$$

The checkers calculus is **confluent** but does not preserve interaction steps.

We consider the **head interaction cost** :

$\mathfrak{t} \Downarrow_{\mathtt{h}}^{\bullet k}$ means $\mathfrak{t}$ $\mathtt{head}$-normalizes with k interaction steps

Counting Interactions in Contextual Equivalence

Let $\mathfrak{t}, \mathfrak{t}' \in \Lambda_{\bullet\circ}$

- **Quantitative Contextual Equivalence:**
  $\mathfrak{t} \equiv^{\mathrm{ctx}}_{◐} \mathfrak{u}$ if,
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Key Point: we ignore silent steps and count interaction steps.

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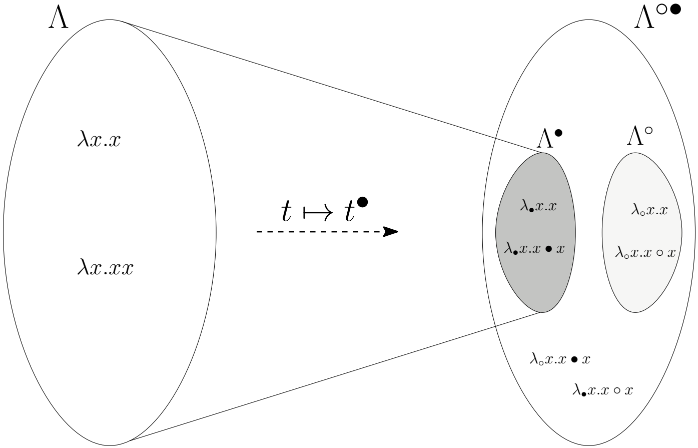
  $\mathfrak{t} \sqsubseteq^{\mathrm{ctx}}_{\bullet} \mathfrak{u}$ [$\mathfrak{u}$ simulates $\mathfrak{t}$] if,
  $$\forall \text{ colored } \mathfrak{C}, \forall k, \quad [\ \mathfrak{C}\langle \mathfrak{t} \rangle \Downarrow^{\bullet k}_{\mathtt{h}} \implies \mathfrak{C}\langle \mathfrak{u} \rangle \Downarrow^{\bullet k}_{\mathtt{h}}\ ];$$

- **Quantitative Contextual Improvement:**
  $\mathfrak{t} \sqsubseteq^{\mathrm{int \cdot imp}} \mathfrak{u}$ [$\mathfrak{u}$ improves $\mathfrak{t}$] if,
  $$\forall \text{ colored } \mathfrak{C}, \forall k, \quad [\ \mathfrak{C}\langle \mathfrak{t} \rangle \Downarrow^{\bullet k}_{\mathtt{h}} \implies \mathfrak{C}\langle \mathfrak{u} \rangle \Downarrow^{\bullet k'}_{\mathtt{h}} \text{ with } k' \leq k\ ];$$

Key Point: we ignore silent steps and count interaction steps.

Interaction Equivalence



$t \sqsubseteq^{\mathrm{int}} u$ if $t^{\bullet} \sqsubseteq^{\mathrm{ctx}}_{\LEFTcircle} u^{\bullet}$ $\quad \ldots$

Note that contexts have both **black or white constructors**.

But... What terms are interaction (in)equivalent?

η-equivalence: $\lambda x.tx =_\eta t$ if $x \notin \mathtt{fv}(t)$

$$\mathtt{I} := \lambda x.x \not\equiv^{\mathrm{int}} \lambda x.\lambda y.xy =: \mathtt{1}$$

- The **separating** context $\mathfrak{C} := \langle\cdot\rangle \circ z \circ w$ distinguishes $\mathtt{1}$ and $\mathtt{I}$:

$$\begin{tikzcd}
\mathtt{1}_\bullet \circ z \circ w \arrow[r, equal, "\bullet\eta"] \arrow[d, "\mathtt{h}_{\odot}"] & \mathtt{I}_\bullet \circ z \circ w \arrow[d, "\mathtt{h}_{\odot}"] \\
(\lambda_\bullet y.z \bullet y) \circ w \arrow[d, "\mathtt{h}_{\odot}"] & z \circ w \\
z \bullet w &
\end{tikzcd}$$

η-reduction may improve terms, but η-expanding them can never lead to improvement:

$$\lambda x.\lambda y.xy \sqsubseteq^{\mathrm{int\cdot imp}} \lambda x.x \qquad \text{but} \qquad \lambda x.x \not\sqsubseteq^{\mathrm{int\cdot imp}} \lambda x.\lambda y.xy$$

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Interaction Equivalence $\iff$ Böhm[POPL 2025]

The interaction preorder is characterized by the preorder induced by Böhm trees $\sqsubseteq_{\mathcal{B}}$.

Intersection Types

$$\begin{tikzcd}
t \equiv_{\mathcal{B}} u \arrow[r, leftrightarrow, "\text{In the literature}"] & t \equiv^{\mathrm{rel}} u & \\
t \sqsubseteq_{\mathcal{B}} u \arrow[r, bend left, "\text{Quantitative properties}"] & t^{\bullet} \sqsubseteq^{\mathrm{rel}}_{\LEFTcircle} u^{\bullet} \arrow[r, bend left, "\text{Tight technique}"] & t \sqsubseteq^{\mathrm{int}} u \arrow[ll, bend left, "\text{Interaction Böhm out technique}"]
\end{tikzcd}$$

Characterizing Interaction Improvement [FoSSaCS 2026]

$$
\begin{tikzcd}
 & t \sqsubseteq^{\mathrm{rel}} u \arrow[dl, bend right, leftrightarrow, "\text{In the literature}"'] & \\
t \sqsubseteq_{\mathcal{B}+\eta red} u \arrow[r, dotted, -] & t^{\bullet} \overset{?}{\sqsubseteq}{}^{\mathrm{rel}}_{☯} u^{\bullet} \arrow[r, dotted] & t \sqsubseteq^{\mathrm{int\cdot imp}} u \arrow[ll, bend left, "\text{Interaction Böhm out technique}"]
\end{tikzcd}
$$

Studied through Intersection Types

Intersection Types for the Plain λ-Calculus

A syntactic presentation of $[\![t]\!]_{\mathrm{rel}}$.

Terms can multiple types:
$x : \beta_1 \cap \beta_2, \ldots, y : \alpha \vdash t : \alpha \cap \beta$

All terminating terms are typable:
$x : \beta \to \alpha, x : \beta \vdash xx : \alpha$

The application rule of intersection types:

$$\frac{\Gamma \vdash t : [\beta_1, \ldots, \beta_i] \multimap \alpha \quad \Delta_1 \vdash u : \beta_1 \quad \cdots \quad \Delta_i \vdash u : \beta_i}{\Gamma \uplus \Delta_1 \uplus \cdots \uplus \Delta_i \vdash tu : \alpha} @$$

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Intersection Types for the Checkers Calculus

$$\begin{array}{llll} \textsc{Types} & \mathsf{L}, \mathsf{L}' & ::= & \mathsf{A} \mid \mathsf{M} \xrightarrow{\mathsf{c}} \mathsf{L} & \mathsf{c} \in \{\circ, \bullet\} \\ \textsc{Multi types} & \mathsf{M}, \mathsf{N} & ::= & [\mathsf{L}_1, \ldots, \mathsf{L}_n] & n \geq 0 \end{array}$$

$$\frac{}{x : [\mathsf{L}] \vdash^{0} x : \mathsf{L}}\ \mathsf{ax} \qquad \frac{(\Gamma_i \vdash^{l_i} t : \mathsf{L}_i)_{i \in I} \quad I \text{ finite}}{\uplus_{i \in I} \Gamma_i \vdash^{\sum_{i \in I} l_i} t : [\mathsf{L}_i]_{i \in I}}\ \mathsf{many}$$

$$\frac{\Gamma, x : \mathsf{M} \vdash^{k} t : \mathsf{L}}{\Gamma \vdash^{k} \lambda_{\mathsf{c}} x.t : \mathsf{M} \xrightarrow{\mathsf{c}} \mathsf{L}}\ \lambda$$

$$\frac{\Gamma \vdash^{l} t : \mathsf{M} \xrightarrow{\mathsf{c}} \mathsf{L}' \quad \Delta \vdash^{h} u : \mathsf{M}}{\Gamma \uplus \Delta \vdash^{l+h+1} t \cdot^{\mathsf{c}^{\perp}} u : \mathsf{L}}\ @_{\bullet} \qquad \frac{\Gamma \vdash^{l} t : \mathsf{M} \xrightarrow{\mathsf{c}} \mathsf{L}' \quad \Delta \vdash^{h} u : \mathsf{M}}{\Gamma \uplus \Delta \vdash^{l+h} t \cdot^{\mathsf{c}} u : \mathsf{L}}\ @_{\tau}$$

As always:

- Typability=Termination: $t \Downarrow_{\mathrm{h}} \iff \exists \Gamma, \mathsf{L}, \Gamma \vdash t : \mathsf{L}$
- Interaction Upper Bound: $\exists \Gamma, \mathsf{L}, \Gamma \vdash^{k'} t : \mathsf{L} \Rightarrow t \Downarrow_{\mathrm{h}}^{\bullet k}$ s.t. $k' \geq k$

Intersection Types for the Checkers Calculus

$$\begin{array}{lrcll}
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\end{array}$$

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Intersection Types for the Checkers Calculus

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$$

$$
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$$

$$
\frac{\Gamma, x : \mathsf{M} \vdash^{k}_{\bullet} t : \mathsf{L}}{\Gamma \vdash^{k}_{\bullet} \lambda_{\mathsf{c}} x. t : \mathsf{M} \xrightarrow{\mathsf{c}} \mathsf{L}}\ \lambda
$$

$$
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Relational Checkers Semantics

$$[\![t]\!]_{\mathrm{rel}\LEFTcircle} = \{(\Gamma, \mathsf{L}, k) \mid \Gamma \vdash^{k}_{\LEFTcircle} t : \mathsf{L}\}$$

$$[\![t]\!]_{\mathrm{rel}\LEFTcircle} = [\![u]\!]_{\mathrm{rel}\LEFTcircle} \iff \mathcal{B} \iff t \equiv^{\mathrm{int}} u \iff t \equiv_{\mathrm{rel}} u$$ [POPL25]

$$[\![t]\!]_{\mathrm{rel}\LEFTcircle} \subseteq [\![u]\!]_{\mathrm{rel}\LEFTcircle} \not\Longleftrightarrow t \sqsubseteq^{\mathrm{int\cdot imp}} u$$

$$\vdash^{1}_{\LEFTcircle} \lambda_{\bullet} x.\lambda_{\bullet} y.x \bullet y : [\mathbf{0} \xrightarrow{\circ} \mathsf{L}] \xrightarrow{\bullet} \mathbf{0} \xrightarrow{\bullet} \mathsf{L}$$

whereas

$$\vdash^{0}_{\LEFTcircle} \lambda_{\bullet} x.x : [\mathbf{0} \xrightarrow{\mathsf{c}} \mathsf{L}] \xrightarrow{\bullet} \mathbf{0} \xrightarrow{\mathsf{c}} \mathsf{L}$$

Candidate: $[\![\lambda_{\bullet} x.\lambda_{\bullet} y.x \bullet y]\!]_{\mathrm{rel}\LEFTcircle} \subseteq^{\text{whiter cheaper}} [\![\lambda_{\bullet} x.x]\!]_{\mathrm{rel}\LEFTcircle}$

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Difficulty: Proving Compatibility

App Case: If $t \sqsubseteq^{\text{whiter cheaper}} u$ and $s \sqsubseteq^{\text{whiter cheaper}} r$ then $t \cdot^{\mathsf{c}} s \sqsubseteq^{\text{whiter cheaper}} u \cdot^{\mathsf{c}} r$ for any c.

$$\frac{\Gamma_1 \vdash^{k_1}_{\bullet} t : \mathsf{M} \xrightarrow{\mathsf{d}} \mathsf{L} \quad \Gamma_2 \vdash^{k_2}_{\bullet} s : \mathsf{M}}{\Gamma \uplus \Delta \vdash^{k_1+k_2+\delta^{\perp}_{\mathsf{c},\mathsf{d}}}_{\bullet} t \cdot^{\mathsf{c}} s : \mathsf{L}} \ @$$

By hypothesis (**whiter cheaper** premices):

$$\frac{\Gamma'_1 \vdash^{k'_1}_{\bullet} u : \mathsf{M}' \xrightarrow{\mathsf{d}'} \mathsf{L}' \quad \Gamma'_2 \vdash^{k'_2}_{\bullet} r : \mathsf{M}''}{\Gamma'_1 \uplus \Gamma'_2 \vdash^{k'_1+k'_2+\delta^{\perp}_{\mathsf{c},\mathsf{d}'}}_{\bullet} u \cdot^{\mathsf{c}} r : \mathsf{L}'} \ @??(\text{if } \mathsf{M}' = \mathsf{M}'')$$

Both are **whiter** versions of M, we *repaint* typings **whiter** to be able to apply @.

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Both are **whiter** versions of M, we *repaint* typings **whiter** to be able to apply @.

Example of Repainting

$$\cfrac{x\!:\![\mathsf{M} \xrightarrow{\circ} \mathsf{L}] \vdash^{1}_{◐} \lambda_{\bullet} y.x \bullet y : \mathsf{M} \xrightarrow{\bullet} \mathsf{L} \quad \Gamma \vdash^{k}_{◐} s : \mathsf{M}}{\Gamma, x\!:\![\mathsf{M} \xrightarrow{\circ} \mathsf{L}] \vdash^{1+k+\delta^{\perp}_{\bullet,\mathsf{d}}}_{◐} (\lambda_{\bullet} y.x \bullet y) \cdot^{\mathsf{d}} s : \mathsf{L}} \;@$$

$$\downarrow (\lambda_{\bullet} y.x \bullet y \sqsubseteq^{\text{whiter cheaper}} x)$$

$$\cfrac{x\!:\![\mathsf{M} \xrightarrow{\circ} \mathsf{L}] \vdash^{0}_{◐} x : \mathsf{M} \xrightarrow{\circ} \mathsf{L} \quad \Gamma \vdash^{k}_{◐} s : \mathsf{M}}{\Gamma, x\!:\![\mathsf{M} \xrightarrow{\circ} \mathsf{L}] \vdash^{k+\delta^{\perp}_{\circ,\mathsf{d}}}_{◐} x \cdot^{\mathsf{d}} s : \mathsf{L}} \;@$$

$$\downarrow (s \sqsubseteq^{\text{whiter cheaper}} r)$$

$$\cfrac{x\!:\![\mathsf{M}' \xrightarrow{\circ} \mathsf{L}] \vdash^{0}_{◐} x : \mathsf{M}' \xrightarrow{\circ} \mathsf{L} \quad \Gamma' \vdash^{k'}_{◐} r : \mathsf{M}'}{\Gamma', x\!:\![\mathsf{M}' \xrightarrow{\circ} \mathsf{L}] \vdash^{k'+\delta^{\perp}_{\circ,\mathsf{d}}}_{◐} x \cdot^{\mathsf{d}} s : \mathsf{L}} \;@$$

Example of Repainting

$$\dfrac{x : [\mathsf{M} \xrightarrow{\circ} \mathsf{L}] \vdash^{1}_{\LEFTcircle} \lambda_{\bullet} y. x \bullet y : \mathsf{M} \xrightarrow{\bullet} \mathsf{L} \quad \Gamma \vdash^{k}_{\LEFTcircle} s : \mathsf{M}}{\Gamma, x : [\mathsf{M} \xrightarrow{\circ} \mathsf{L}] \vdash^{1+k+\boldsymbol{\delta}^{\perp}_{\bullet,\mathsf{d}}}_{\LEFTcircle} (\lambda_{\bullet} y. x \bullet y) \cdot^{\mathsf{d}} s : \mathsf{L}} \ \texttt{@}$$

$$\downarrow (\lambda_{\bullet} y. x \bullet y \ \sqsubseteq^{\text{whiter cheaper}} \ x)$$

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$$\downarrow (s \ \sqsubseteq^{\text{whiter cheaper}} \ r)$$

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Whiter Cheaper but Not Everywhere...

$$(x : [\mathsf{M} \xrightarrow{\mathsf{c}} \mathsf{L}], \mathsf{M} \xrightarrow{\mathsf{d}} \mathsf{L}, k) \in [\![\lambda_\bullet y.x \bullet y]\!]^{◐}$$

iff

$$\mathsf{d} = \bullet \text{ and } k = \begin{cases} 0 & \text{if } \mathsf{c} = \mathsf{d} \\ 1 & \text{if } \mathsf{c} \neq \mathsf{d} \end{cases}$$

The **whiter and cheaper** typing whitens in specific spots:

$$(x : [\mathsf{M} \xrightarrow{\circ} \mathsf{L}], \mathsf{M} \xrightarrow{\circ} \mathsf{L}, 0) \in [\![x]\!]^{◐}$$

Whiter Cheaper but Not Everywhere...

$$(x : [\mathsf{M} \xrightarrow{\mathsf{c}} \mathsf{L}], \mathsf{M} \xrightarrow{\mathsf{d}} \mathsf{L}, k) \in [\![\lambda_\bullet y.x \bullet y]\!]^{\yinyang}$$

$$\text{iff}$$

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The **whiter and cheaper** typing whitens in specific spots:

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Polarized Whiter Cheaper

For all checkers terms $\mathfrak{t}$, $\mathfrak{u}$ we define $\mathfrak{t} \sqsubseteq^{\mathrm{pwc}}_{\bullet} \mathfrak{u}$ if and only if

$$\begin{gathered}\forall(\Gamma, \mathsf{L}, k), \text{ if } \Gamma \vdash^{k}_{\bullet} \mathfrak{t} : \mathsf{L}, \text{ then} \\ \exists(\Gamma', \mathsf{L}', k') \text{ such that } \Gamma' \vdash^{k'}_{\bullet} \mathfrak{u} : \mathsf{L}' \\ \text{and } \exists d, \langle \Gamma', \mathsf{L}' \rangle \leq^{+}_{d} \langle \Gamma, \mathsf{L} \rangle \text{ with } k \geq k' + d\end{gathered}$$

Polarized Color Occurrences

$$\frac{\mathsf{p} \in \{+, -\}}{\mathsf{A} \leq_0^{\mathsf{p}} \mathsf{A}} \qquad \frac{\mathsf{M}' \leq_{k_1}^{-} \mathsf{M} \quad \mathsf{L}' \leq_{k_2}^{+} \mathsf{L}}{\mathsf{M}' \xrightarrow{\circ} \mathsf{L}' \leq_{k_1+k_2+1}^{+} \mathsf{M} \xrightarrow{\bullet} \mathsf{L}} \qquad \frac{\mathsf{M}' \leq_{k_1}^{\neg\mathsf{p}} \mathsf{M} \quad \mathsf{L}' \leq_{k_2}^{\mathsf{p}} \mathsf{L}}{\mathsf{M}' \xrightarrow{\mathsf{c}} \mathsf{L}' \leq_{k_1+k_2}^{\mathsf{p}} \mathsf{M} \xrightarrow{\mathsf{c}} \mathsf{L}}$$

$$\frac{\mathsf{L}_1' \leq_{k_1}^{\mathsf{p}} \mathsf{L}_1 \quad \cdots \quad \mathsf{L}_n' \leq_{k_n}^{\mathsf{p}} \mathsf{L}_n}{[\mathsf{L}_1', \ldots, \mathsf{L}_n'] \leq_{k_1+\cdots+k_n}^{\mathsf{p}} [\mathsf{L}_1, \ldots, \mathsf{L}_n]}$$

Figure: Polarized whitening of a type.

The relations above extend to environments Γ and pairs $\langle \Gamma, \mathsf{L} \rangle$ as follows:

$$\frac{}{\emptyset \leq_0^{\mathsf{p}} \emptyset} \qquad \frac{\Gamma' \leq_{k_1}^{\mathsf{p}} \Gamma \quad \mathsf{M}' \leq_{k_2}^{\mathsf{p}} \mathsf{M}}{\Gamma', x : \mathsf{M}' \leq_{k_1+k_2}^{\mathsf{p}} \Gamma, x : \mathsf{M}} \qquad \frac{\Gamma' \leq_{k_1}^{\neg\mathsf{p}} \Gamma \quad \mathsf{L}' \leq_{k_2}^{\mathsf{p}} \mathsf{L}}{\langle \Gamma', \mathsf{L}' \rangle \leq_{k_1+k_2}^{\mathsf{p}} \langle \Gamma, \mathsf{L} \rangle}$$

Compatibility of the Polarized Whiter Cheaper Improvement

App Case: If $t \sqsubseteq^{\mathrm{pwc}}_{\bullet} u$ and $s \sqsubseteq^{\mathrm{pwc}}_{\bullet} r$ then $t \cdot^{\mathsf{c}} s \sqsubseteq^{\mathrm{pwc}}_{\bullet} u \cdot^{\mathsf{c}} r$ for any c.

$$\frac{\Gamma_1 \vdash^{k_1}_{\bullet} t : \mathsf{M} \xrightarrow{\mathsf{d}} \mathsf{L} \quad \Gamma_2 \vdash^{k_2}_{\bullet} s : \mathsf{M}}{\Gamma \uplus \Delta \vdash^{k_1+k_2+\delta^{\perp}_{\mathsf{c},\mathsf{d}}}_{\bullet} t \cdot^{\mathsf{c}} s : \mathsf{L}} \ @$$

By hypothesis ($t \sqsubseteq^{\mathrm{pwc}}_{\bullet} u$ and $s \sqsubseteq^{\mathrm{pwc}}_{\bullet} r$):

$$\begin{array}{c} \Gamma'_1 \vdash^{k'_1}_{\bullet} u : \mathsf{M}' \xrightarrow{\mathsf{d}'} \mathsf{L}' \quad \Gamma'_2 \vdash^{k'_2}_{\bullet} r : \mathsf{M}'' \\ \vdots \\ \Gamma'_1 \uplus \Gamma'_2 \vdash^{k'_1+k'_2+\delta^{\perp}_{\mathsf{c},\mathsf{d}'}}_{\bullet} u \cdot^{\mathsf{c}} r : \mathsf{L}' \end{array} \quad @(\text{if } \mathsf{M}' = \mathsf{M}'')$$

We know that M, M' and M'' are still related:

$$\mathsf{M}' \leq^{-}_{l} \mathsf{M} \text{ and } \mathsf{M}'' \leq^{+}_{h} \mathsf{M}$$

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Compatibility of the Polarized Whiter Cheaper Improvement

App Case: If $t \sqsubseteq_{\bullet}^{\mathrm{pwc}} u$ and $s \sqsubseteq_{\bullet}^{\mathrm{pwc}} r$ then $t \cdot^{\mathsf{c}} s \sqsubseteq_{\bullet}^{\mathrm{pwc}} u \cdot^{\mathsf{c}} r$ for any c.

$$\dfrac{\Gamma_1 \vdash_{\bullet}^{k_1} t : \mathsf{M} \xrightarrow{\mathsf{d}} \mathsf{L} \quad \Gamma_2 \vdash_{\bullet}^{k_2} s : \mathsf{M}}{\Gamma \uplus \Delta \vdash_{\bullet}^{k_1 + k_2 + \boldsymbol{\delta}^{\perp}_{\mathsf{c},\mathsf{d}}} t \cdot^{\mathsf{c}} s : \mathsf{L}} \ @$$

By hypothesis ($t \sqsubseteq_{\bullet}^{\mathrm{pwc}} u$ and $s \sqsubseteq_{\bullet}^{\mathrm{pwc}} r$):

$$\begin{array}{c} \Gamma_1' \vdash_{\bullet}^{k_1'} u : \mathsf{M}' \xrightarrow{\mathsf{d}'} \mathsf{L}' \quad \Gamma_2' \vdash_{\bullet}^{k_2'} r : \mathsf{M}'' \\ \vdots \\ \Gamma_1' \uplus \Gamma_2' \vdash_{\bullet}^{k_1' + k_2' + \boldsymbol{\delta}^{\perp}_{\mathsf{c},\mathsf{d}'}} u \cdot^{\mathsf{c}} r : \mathsf{L}' \end{array} \quad @(\text{if } \mathsf{M}' = \mathsf{M}'')$$

We know that M, M' and M'' are still related:

$$\mathsf{M}' \leq_l^- \mathsf{M} \text{ and } \mathsf{M}'' \leq_h^+ \mathsf{M}$$

Compatibility of the Polarized Whiter Cheaper Improvement

App Case: If $t \sqsubseteq_{\bullet}^{\mathrm{pwc}} u$ and $s \sqsubseteq_{\bullet}^{\mathrm{pwc}} r$ then $t \cdot^{\mathsf{c}} s \sqsubseteq_{\bullet}^{\mathrm{pwc}} u \cdot^{\mathsf{c}} r$ for any c.

$$\dfrac{\Gamma_1 \vdash_{\bullet}^{k_1} t : \mathsf{M} \xrightarrow{\mathsf{d}} \mathsf{L} \quad \Gamma_2 \vdash_{\bullet}^{k_2} s : \mathsf{M}}{\Gamma \uplus \Delta \vdash_{\bullet}^{k_1+k_2+\boldsymbol{\delta}_{\mathsf{c},\mathsf{d}}^{\perp}} t \cdot^{\mathsf{c}} s : \mathsf{L}} \; @$$

By hypothesis ($t \sqsubseteq_{\bullet}^{\mathrm{pwc}} u$ and $s \sqsubseteq_{\bullet}^{\mathrm{pwc}} r$):

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We know that M, M' and M'' are still related:

$$\mathsf{M}' \leq_l^- \mathsf{M} \text{ and } \mathsf{M}'' \leq_h^+ \mathsf{M}$$

Example of Repainting

It does happen that the argument type M is unchanged, consider $\lambda_\bullet y.x \bullet y \sqsubseteq^{\mathrm{pwc}}_{\LEFTcircle} x$ applied to an argument $s \sqsubseteq^{\mathrm{pwc}}_{\LEFTcircle} s$.

$$\dfrac{x : [\mathsf{M} \overset{\circ}{\to} \mathsf{L}] \vdash^{1}_{\LEFTcircle} \lambda_\bullet y.x \bullet y : \mathsf{M} \overset{\bullet}{\to} \mathsf{L} \quad \Gamma \vdash^{k}_{\LEFTcircle} s : \mathsf{M}}{\Gamma, x : [\mathsf{M} \overset{\circ}{\to} \mathsf{L}] \vdash^{1+k+\boldsymbol{\delta}^{\perp}_{\bullet,\mathsf{d}}}_{\LEFTcircle} (\lambda_\bullet y.x \bullet y) \cdot^{\mathsf{d}} s : \mathsf{L}}\ @ \qquad \dfrac{x : [\mathsf{M} \overset{\circ}{\to} \mathsf{L}] \vdash^{0}_{\LEFTcircle} x : \mathsf{M} \overset{\circ}{\to} \mathsf{L} \quad \Gamma \vdash^{k}_{\LEFTcircle} s : \mathsf{M}}{\Gamma, x : [\mathsf{M} \overset{\circ}{\to} \mathsf{L}] \vdash^{k+\boldsymbol{\delta}^{\perp}_{\circ,\mathsf{d}}}_{\LEFTcircle} x \cdot^{\mathsf{d}} s : \mathsf{L}}\ @$$

What if we were to use another term than s as argument, say r such that $s \sqsubseteq^{\mathrm{pwc}}_{\LEFTcircle} r$. We repaint in the typing for x

$$\dfrac{x : [\mathsf{M}' \overset{\circ}{\to} \mathsf{L}] \vdash^{0}_{\LEFTcircle} x : \mathsf{M}' \overset{\circ}{\to} \mathsf{L} \quad \Gamma' \vdash^{k'}_{\LEFTcircle} r : \mathsf{M}'}{\Gamma', x : [\mathsf{M}' \overset{\circ}{\to} \mathsf{L}] \vdash^{k'+\boldsymbol{\delta}^{\perp}_{\circ,\mathsf{d}}}_{\LEFTcircle} x \cdot^{\mathsf{d}} s : \mathsf{L}}\ @$$

Consider $\lambda_\bullet x.x \bullet \lambda_\bullet z.y \bullet z \sqsubseteq^{\mathrm{pwc}}_{\LEFTcircle} \lambda_\bullet x.x \bullet y$ applied to s.
We have a derivation for $(\lambda_\bullet x.x \bullet \lambda_\bullet z.y \bullet z) \cdot^{\mathsf{d}} s$:

$$
\dfrac{
\dfrac{
\dfrac{
x : [[\mathsf{M} \xrightarrow{\bullet} \mathsf{L}] \xrightarrow{\mathsf{c}} \mathsf{L}'] \vdash^{0}_{\LEFTcircle} x : [\mathsf{M} \xrightarrow{\bullet} \mathsf{L}] \xrightarrow{\mathsf{c}} \mathsf{L}' \quad y : [\mathsf{M} \xrightarrow{\circ} \mathsf{L}] \vdash^{1}_{\LEFTcircle} \lambda_\bullet z.y \bullet z : [\mathsf{M} \xrightarrow{\bullet} \mathsf{L}]
}{
y : [\mathsf{M} \xrightarrow{\circ} \mathsf{L}], x : [[\mathsf{M} \xrightarrow{\bullet} \mathsf{L}] \xrightarrow{\mathsf{c}} \mathsf{L}'] \vdash^{1+\delta^{\perp}_{\mathsf{c},\bullet}}_{\LEFTcircle} x \bullet \lambda_\bullet z.y \bullet z : \mathsf{L}'
} \; @
}{
y : [\mathsf{M} \xrightarrow{\circ} \mathsf{L}] \vdash^{1+\delta^{\perp}_{\mathsf{c},\bullet}}_{\LEFTcircle} \lambda_\bullet x.x \bullet \lambda_\bullet z.y \bullet z : [[\mathsf{M} \xrightarrow{\bullet} \mathsf{L}] \xrightarrow{\mathsf{c}} \mathsf{L}'] \xrightarrow{\bullet} \mathsf{L}'
} \; \lambda
\qquad \Gamma \vdash^{k}_{\LEFTcircle} s : [[\mathsf{M} \xrightarrow{\bullet} \mathsf{L}] \xrightarrow{\mathsf{c}} \mathsf{L}']
}{
\Gamma, y : [\mathsf{M} \xrightarrow{\circ} \mathsf{L}] \vdash^{1+k+\delta^{\perp}_{\bullet,\mathsf{d}}+\delta^{\perp}_{\mathsf{c},\bullet}}_{\LEFTcircle} (\lambda_\bullet x.x \bullet \lambda_\bullet z.y \bullet z) \cdot^{\mathsf{d}} s : \mathsf{L}'
} \; @
$$

There exists a positive whiter-cheaper derivation for $\lambda_\bullet x.x \bullet y$:

$$
\dfrac{
x : [[\mathsf{M} \xrightarrow{\circ} \mathsf{L}] \xrightarrow{\mathsf{c}} \mathsf{L}'] \vdash^{0}_{\LEFTcircle} x : [\mathsf{M} \xrightarrow{\circ} \mathsf{L}] \xrightarrow{\mathsf{c}} \mathsf{L}' \quad y : [\mathsf{M} \xrightarrow{\circ} \mathsf{L}] \vdash^{0}_{\LEFTcircle} y : [\mathsf{M} \xrightarrow{\circ} \mathsf{L}]
}{
\dfrac{
y : [\mathsf{M} \xrightarrow{\circ} \mathsf{L}], x : [[\mathsf{M} \xrightarrow{\circ} \mathsf{L}] \xrightarrow{\mathsf{c}} \mathsf{L}'] \vdash^{\delta^{\perp}_{\mathsf{c},\bullet}}_{\LEFTcircle} x \bullet y : \mathsf{L}'
}{
y : [\mathsf{M} \xrightarrow{\circ} \mathsf{L}] \vdash^{\delta^{\perp}_{\mathsf{c},\bullet}}_{\LEFTcircle} \lambda_\bullet x.x \bullet y : [[\mathsf{M} \xrightarrow{\circ} \mathsf{L}] \xrightarrow{\mathsf{c}} \mathsf{L}'] \xrightarrow{\bullet} \mathsf{L}'
} \; \lambda
} \; @
\qquad \Gamma \vdash^{k}_{\LEFTcircle} s : [[\mathsf{M} \xrightarrow{\bullet} \mathsf{L}] \xrightarrow{\mathsf{c}} \mathsf{L}']
$$

$$
\vdots \; @?
$$

$$
\Gamma, y : [\mathsf{M} \xrightarrow{\circ} \mathsf{L}] \vdash^{k+\delta^{\perp}_{\bullet,\mathsf{d}}+\delta^{\perp}_{\mathsf{c},\bullet}}_{\LEFTcircle} (\lambda_\bullet x.x \bullet y) \cdot^{\mathsf{d}} s : \mathsf{L}'
$$

The repainting mechanism continues on s...

Lemma (Repainting)

Suppose $\Gamma \vdash^{k}_{\bullet} t : \mathsf{L}$ *and that* $\langle \Gamma', \mathsf{L}' \rangle \leq^{-}_{1} \langle \Gamma, \mathsf{L} \rangle$. *Then there exists a derivation of* $\Gamma'' \vdash^{k'}_{\bullet} t : \mathsf{L}''$ *such that one of the following holds:*

(i) $\langle \Gamma'', \mathsf{L}'' \rangle \leq^{+}_{1} \langle \Gamma', \mathsf{L}' \rangle$ *and* $k = k'$*; or*

(ii) $\langle \Gamma'', \mathsf{L}'' \rangle = \langle \Gamma', \mathsf{L}' \rangle$ *and* $|k' - k| = 1$.

Conclusion

- [POPL25] Interaction-awareness in the λ-calculus: checkers calculus & interaction equivalence
- [FoSSaCS26] **Interaction improvement**: make explicit the costs at play in relational semantics
- A new proof technique for compatibility for non standard preorder on intersection type semantics

Future work:

- Work it out in Call-by-Value/Need, and in effectful extensions
  $\lambda f.f\,() \equiv_{\mathrm{ctx}} \lambda f.f\,()\,;f\,()$
- How does it relate to Game Semantics? to process calculi?
- Prove that compiler transformations are correct *improvements*
- What does our interaction cost represent?
  Communication complexity?

Thank you!

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Thank you!

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