A Simple Formal Proof of the Genericity Lemma

Adrienne Lancelot, University of Bologna

Based on joint works with Beniamino Accattoli and Maxime Vemclefs:

- *Light Genericity*, FoSSaCS 2024
- *Barendregt's Theory of the λ-Calculus, Refreshed and Formalized*, ITP 2025
- My PhD thesis, 2025

June 25th 2026 - Chocola seminar

Partial Recursive Functions

Partial Recursive Functions model which mathematical functions are computable.

There is a natural *extensional preorder* on partial functions

$$f \leq_{\text{prf}} g \text{ if } \forall n \in \mathbb{N},\ f(n) = \bot \text{ or } f(n) =_{\mathbb{N}} g(n)$$

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Lambda Calculus

PRF do not look at how to compute, hence the preorder can only be extensional.

Instead, in the lambda calculus, how to compute is a critical concept.

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Computable Functions & Lambda Calculus

Partial recursive functions embed in the lambda calculus.

What is the lambda term that represents **undefined**?

A computation that never ends? Ω!

$\Omega := (\lambda x.xx)(\lambda x.xx) \rightarrow_\beta (\lambda x.xx)(\lambda x.xx) \rightarrow_\beta \cdots$

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A first naive attempt

Undefined represents a computation that never ends.

- **undefined** terms $=$ β-diverging terms?

Surprisingly, this would lead to an inconsistency.

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β-diverging terms may be very different

Indeed, let us look at two β-diverging terms

$$
\begin{array}{ccc}
\mathit{fix} & \text{and} & \Omega \\
\downarrow_\beta & & \downarrow_\beta \\
\lambda f.f\ (\mathit{fix}\, f) & & \Omega \\
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\lambda f.f\ (f\ (f\ (f\ \ldots))) & & \Omega \\
\downarrow_\beta & & \downarrow_\beta \\
\vdots & & \vdots
\end{array}
$$

Recursion does not carry the same meaning as looping on itself.

A second attempt

Instead, one might consider a more restrained reduction

- **undefined** terms = **head**-diverging terms?

The equational theory that identifies **head**-diverging terms is consistent.

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Fixpoint combinators are **head**-normalizing.

$$\begin{array}{ccc} \mathit{fix} & \text{and} & \Omega \\ \downarrow_h & & \downarrow_h \\ \lambda f.f\ (\mathit{fix}\, f) & & \Omega \\ \not\downarrow_h & & \downarrow_h \\ & & \vdots \end{array}$$

Recursion and looping are nicely separated by **head** reduction.

Consistency

A relation $\mathcal{R} \subseteq \Lambda \times \Lambda$ is consistent if there exists $t, u \in \Lambda$ such that $(t, u) \notin \mathcal{R}$.

An equational theory is an equivalence relation $=_{\mathcal{T}}$ such that:

- *Invariance under Computation*: if $t \to_\beta u$ then $t =_{\mathcal{T}} u$
- *Stability by Contexts*: if $t =_{\mathcal{T}} u$ then $\forall C,\ C\langle t \rangle =_{\mathcal{T}} C\langle u \rangle$.

To validate the choice of undefined terms: Is there a consistent equational theory where undefined terms are collapsed?

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What is Genericity?

Undefined terms are black holes for the evaluation process.

If a program awaits the evaluation of an undefined sub-term

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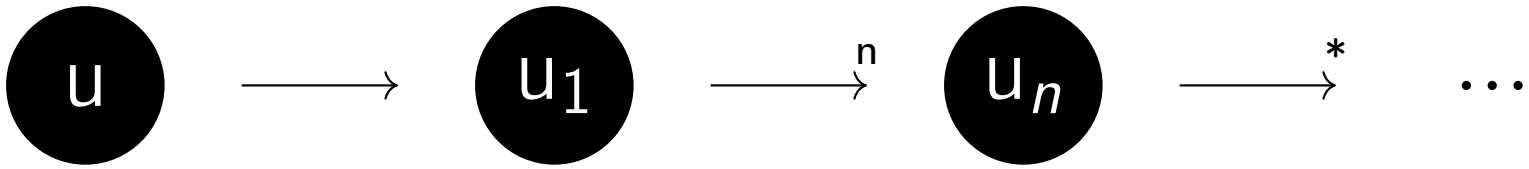

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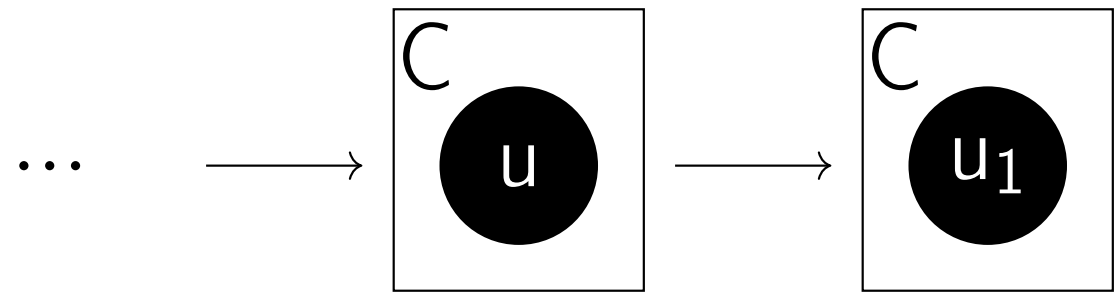

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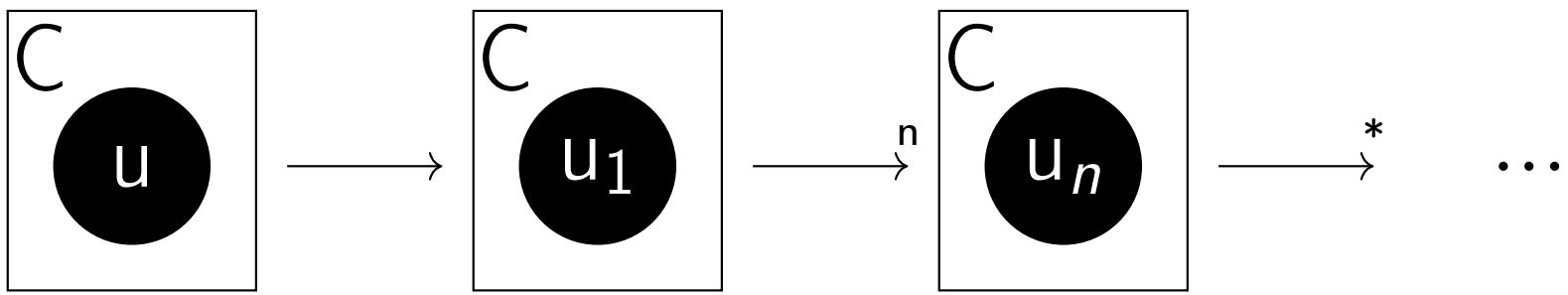

What is Genericity?

Genericity somehow specifies this fact dually:
If a program terminates while there were **undefined** sub-terms, then it never entered the black hole.

Genericity says: (n is a normal form and s is any term)

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Anything can simulate the *generic* **undefined** sub-terms in a terminating term.

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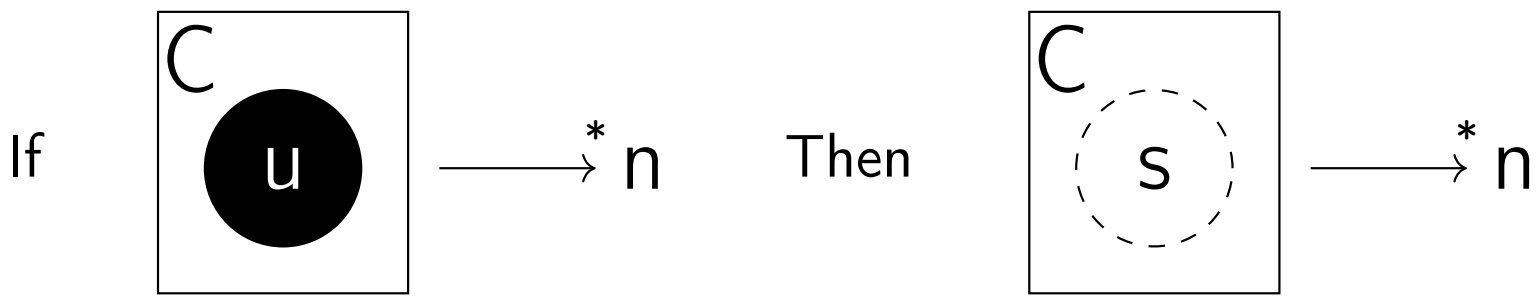

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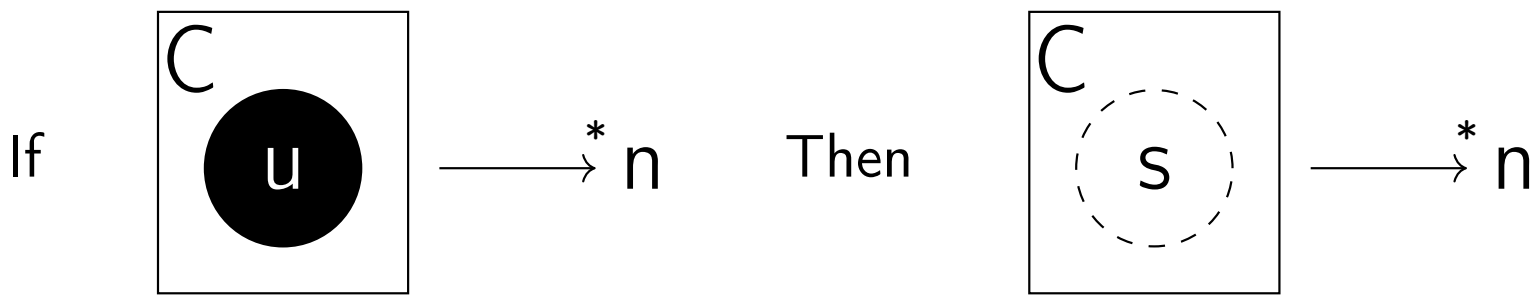

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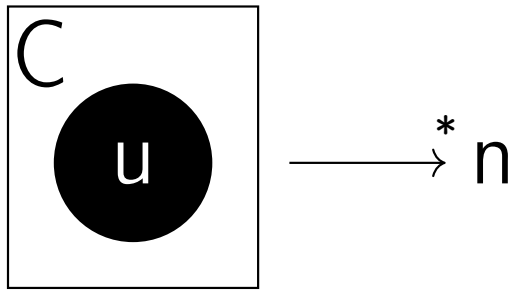
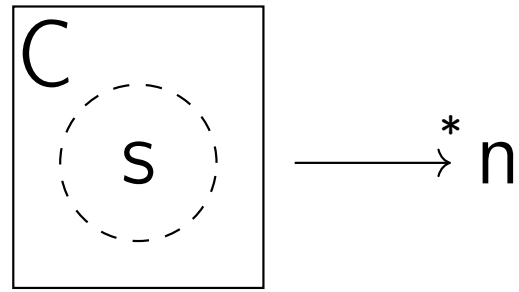
If 

Then 

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Collapsibility: there exists an equational theory $\mathcal{T}$ such that undefined terms are equated in $\mathcal{T}$ and $\mathcal{T}$ is consistent

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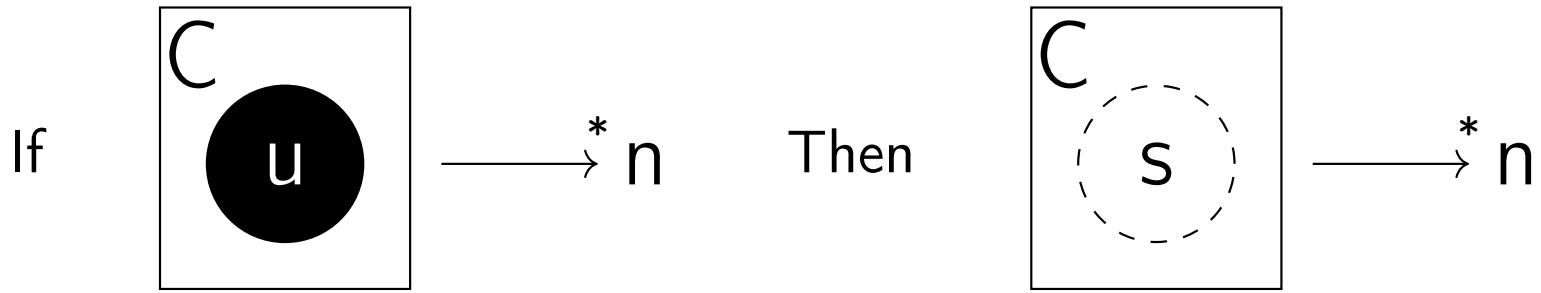

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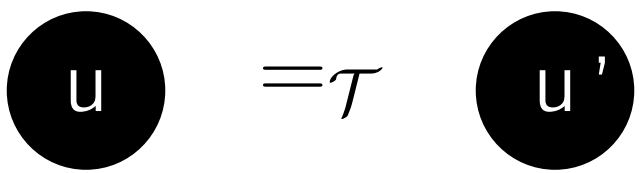

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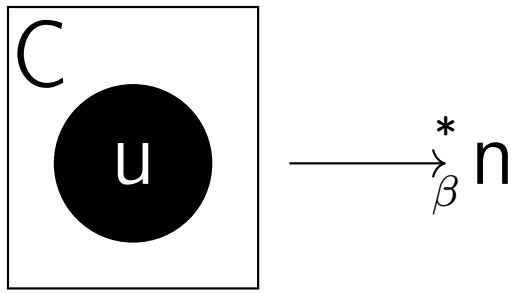
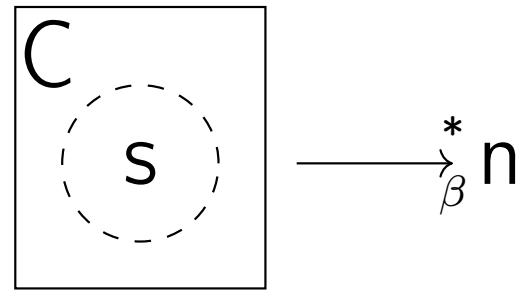
Heavy and Light Genericity

Heavy Genericity: (n is a normal form and s is any term)

If 

Then 

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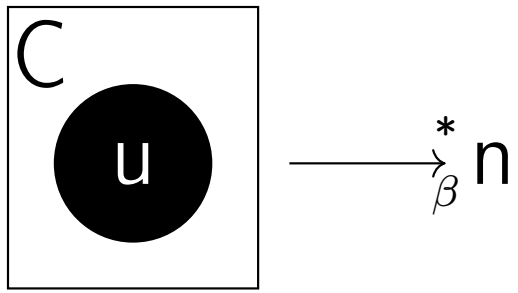
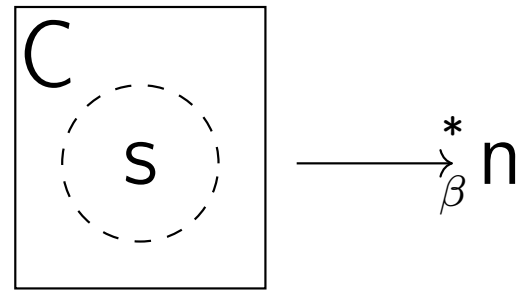
If 

Then 

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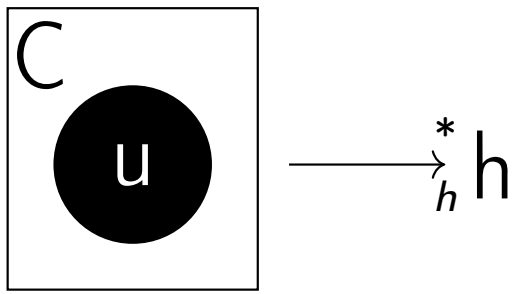
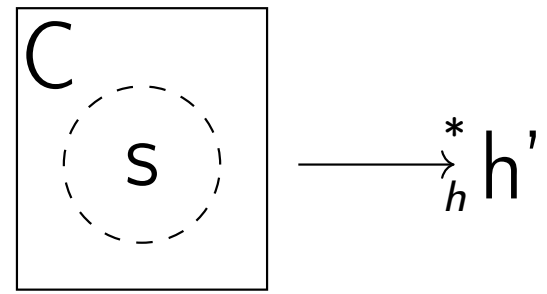
If 

Then 

Genericity is about minimum terms

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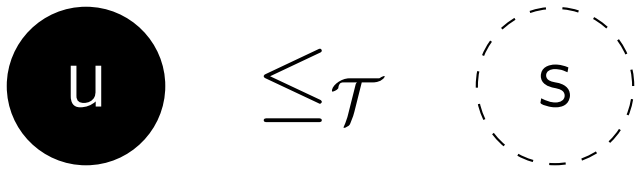

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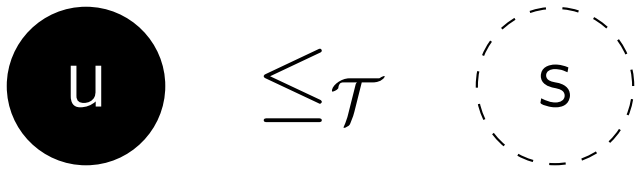

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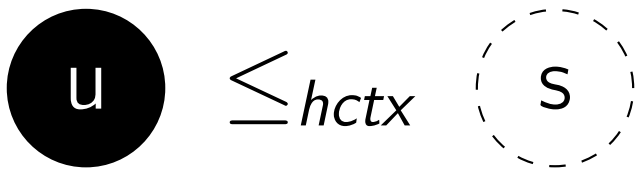

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What I will tell you about today:

- ... are well-known results in Call-by-Name now formalized in a proof assistant
- ... stresses that (Light) Genericity is about minimum terms and contextual preorders

What I won't tell you about today:

- Genericity results for Call-by-Value

  Accattoli-Lancelot 2024; Arrial-Guerrieri-Kesner 2024; ...
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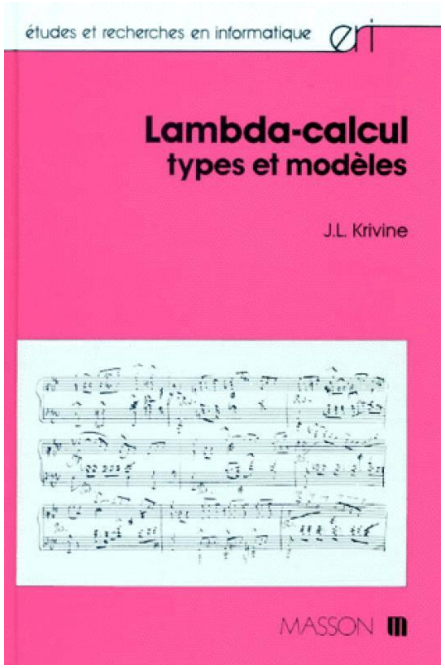

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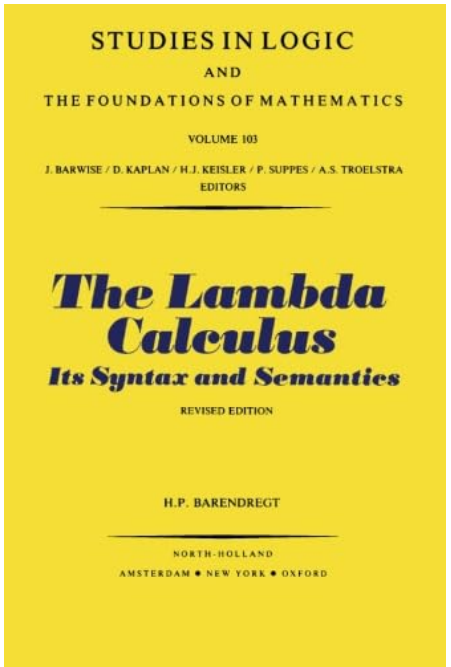

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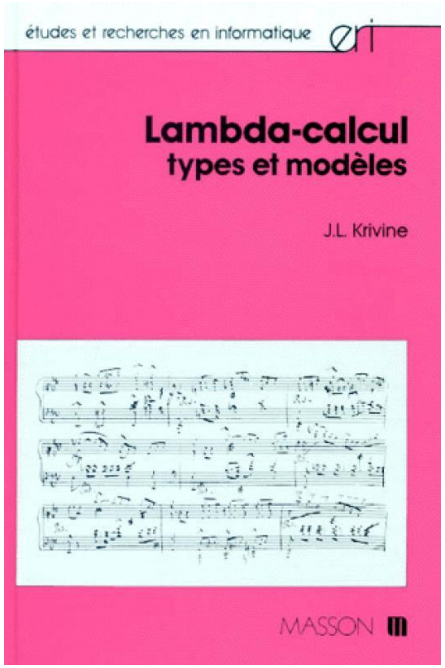

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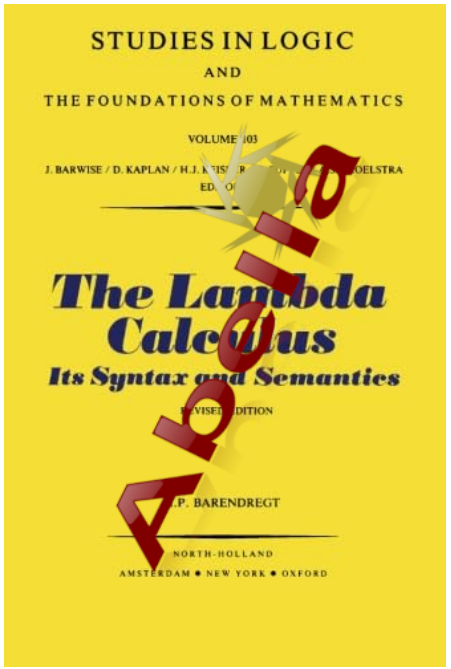

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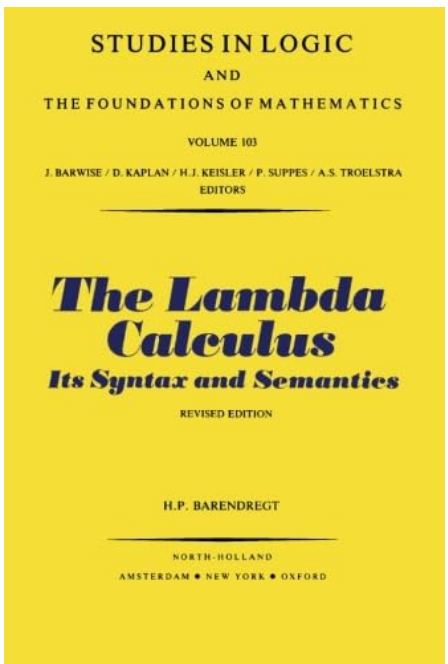

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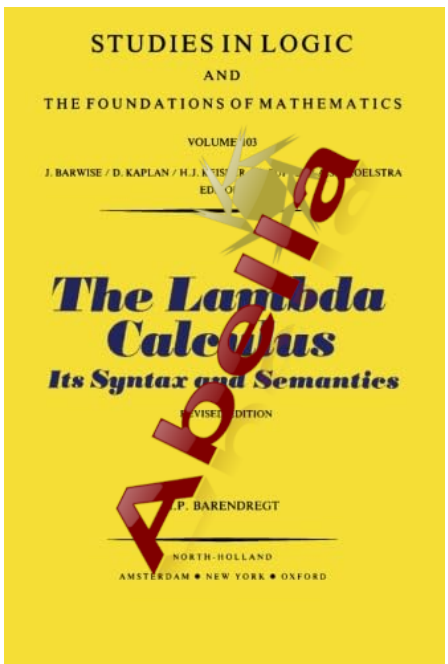

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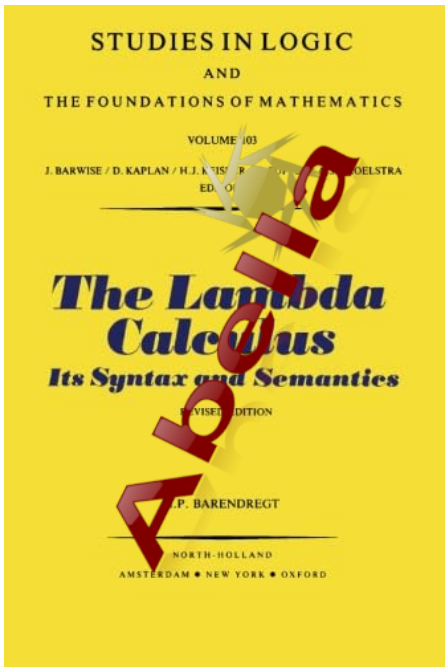

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Proving/Formalizing Genericity

A Simple Proof of the Genericity Lemma

Masako Takahashi

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Tokyo Institute of Technology
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Abstract. A short direct proof is given for the fundamental property of unsolvable λ-terms; if M is an unsolvable λ-term and $C[M]$ is solvable, then $C[N]$ is solvable for any λ-term N. (Here $C[\ ]$ stands for an arbitrary context.)

1. Preliminaries

A term in this note means a λ-term, which is either $x, \lambda x.M$ or MN, (where M, N are terms and x is a variable.) Unless otherwise stated, capital letters $M, N, P, ...$ stand for arbitrary terms, $\mathbf{M}, \mathbf{N}, ...$ for (possibly null) sequences of terms, $x, y, ...$ for variables, and $\mathbf{x}, \mathbf{y}, ...$ for (possibly null) sequences of variables. We refer to [1] as the standard text in the field.

A term of the form $\lambda \mathbf{x}.y\mathbf{M}$ (more precisely, $\lambda x_1.(\lambda x_2.(...(\lambda x_n.((...((yM_1)M_2)...)M_m))...))$ for some $n, m \geq 0$) is said to be in *head normal form* (*hnf*, for short). If a term M has a hnf (that is, $M =_\beta M'$ for a term M' in hnf), then M is called *solvable*. The following are well-known facts of solvable terms (cf.[1] 8.3.1 - 14).

(1) M is solvable if and only if $\forall P, \exists \mathbf{x}, \exists \mathbf{Q}((\lambda \mathbf{x}.M)\mathbf{Q} =_\beta P)$.
(2) $\lambda x.M$ is solvable if and only if so is M.
(3) if $M[x := N]$ is solvable then so is M.
(4) if MN is solvable then so is M.

A term in *β-normal form* (*β-nf*, for short) is recursively defined as a term of the form $\lambda \mathbf{x}.y\mathbf{M}$ where $\mathbf{M}$ is a (possibly null) sequence of terms in β-nf.

2. Propositions

First we prove a special case of the genericity lemma.

Lemma 1. Let M, N, P be terms with M unsolvable and N in β-nf. Then $P[x := M] =_\beta N$ implies $P[x := M'] =_\beta N$ for any M'.

Proof. We prove the lemma by induction on the structure of N. Suppose $P[x := M] =_\beta N$, and $N \equiv \lambda y.zN_1N_2...N_n$ where $n \geq 0$ and each N_i is in β-nf. (Here, $\equiv$ denotes syntactic equality of terms.) Then since N is solvable, P is also solvable by (3) above, and hence has a hnf, say $\lambda u.vP_1P_2...P_p$. Here x and v must be different. (For otherwise $P[x := M] =_\beta \lambda u.M\mathbf{P}$ for some $\mathbf{P}$, and $P[x := M]$ would by (2) and (4) above be unsolvable, which contradicts our assumption.) Therefore we have $P[x := M] =_\beta \lambda u.vP'_1P'_2...P'_p$ where $P'_i \equiv P_i[x := M](i = 1, 2, ..., p)$. Since $P[x := M] =_\beta N \equiv \lambda y.zN_1N_2...N_n$, we know from the Church-Rosser theorem that $P'_i =_\beta N_i (i = 1, 2, ..., n)$ and $p = n$. Without loss of generality we may also assume $\mathbf{u} \equiv \mathbf{y}$ and $v \equiv z$.

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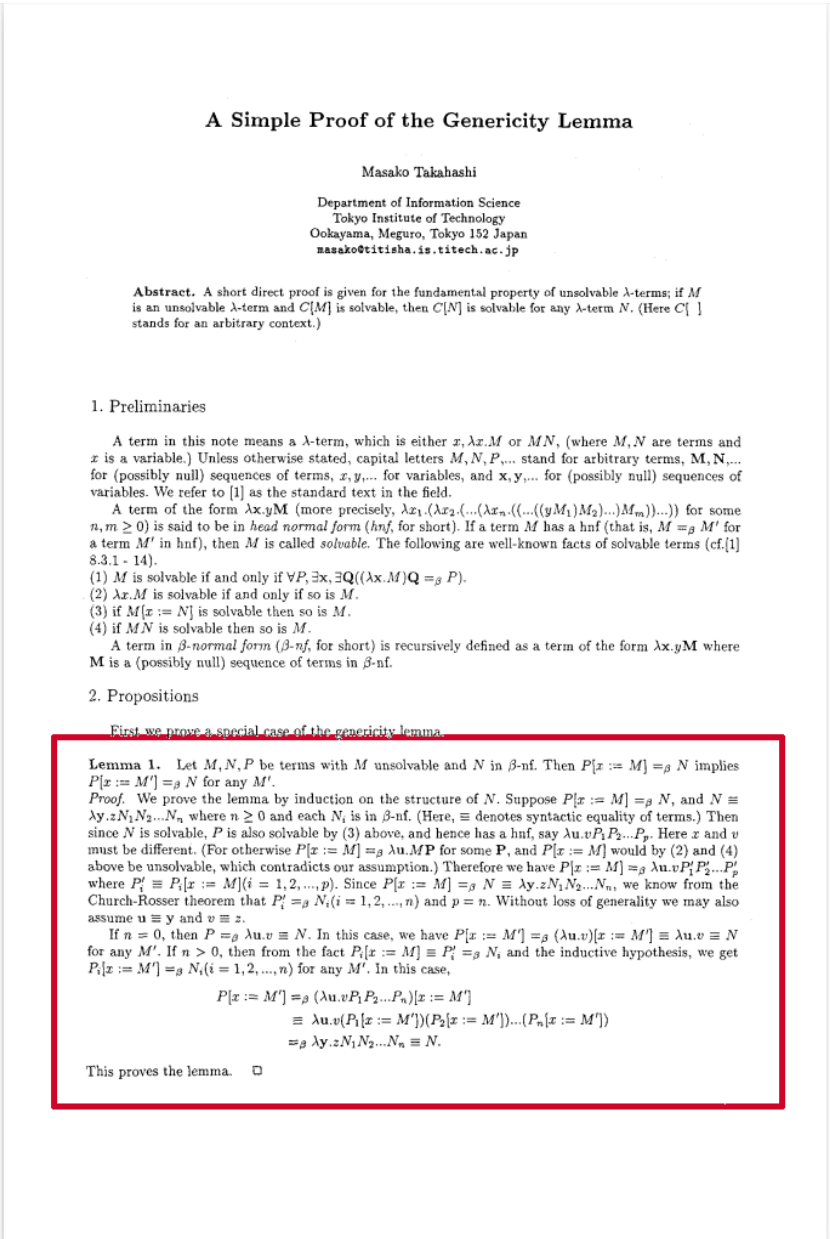
Proving/Formalizing Genericity



A Simple Proof of the Genericity Lemma

Masako Takahashi

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Main Lemma

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~140 tactics

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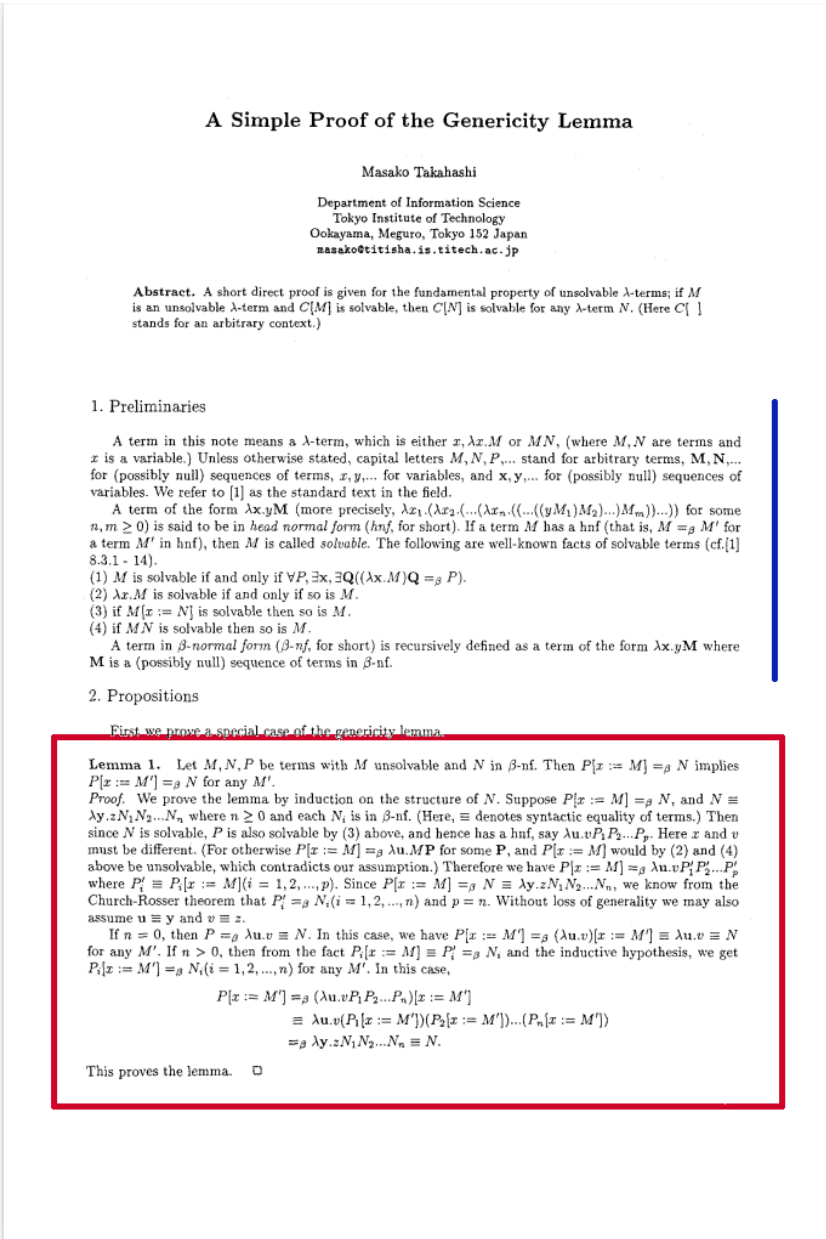

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~15 lines of text
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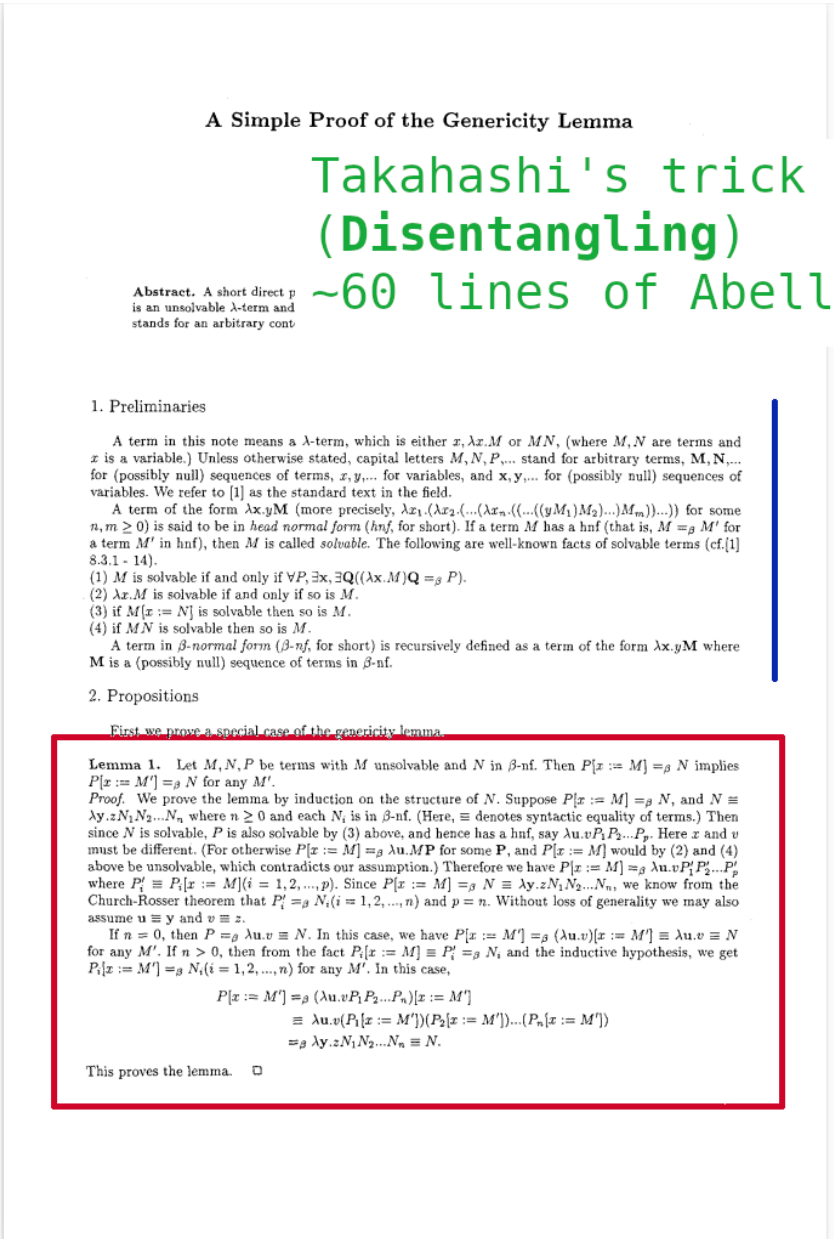

A Simple Proof of the Genericity Lemma

Takahashi's trick
(**Disentangling**)
~60 lines of Abella

Abstract. A short direct p
is an unsolvable λ-term and
stands for an arbitrary cont

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118

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This Talk

- Why is (light) genericity useful?
- Formalizing the theory of the λ-calculus in Abella
- Formalizing Takahashi's proof of genericity

(In)Equational Theories

Fix an evaluation strategy $\mathtt{s}$ within a language of terms $(\mathcal{L}, \rightarrow_{\mathcal{L}})$, $\rightarrow_{\mathtt{s}} \subseteq \rightarrow_{\mathcal{L}}$.

An inequational theory $=_{\mathcal{T}}$ for $\mathtt{s}$ is:

1. an *Equivalence Relation* on terms, $=_{\mathcal{T}} \subseteq \mathcal{L} \times \mathcal{L}$;
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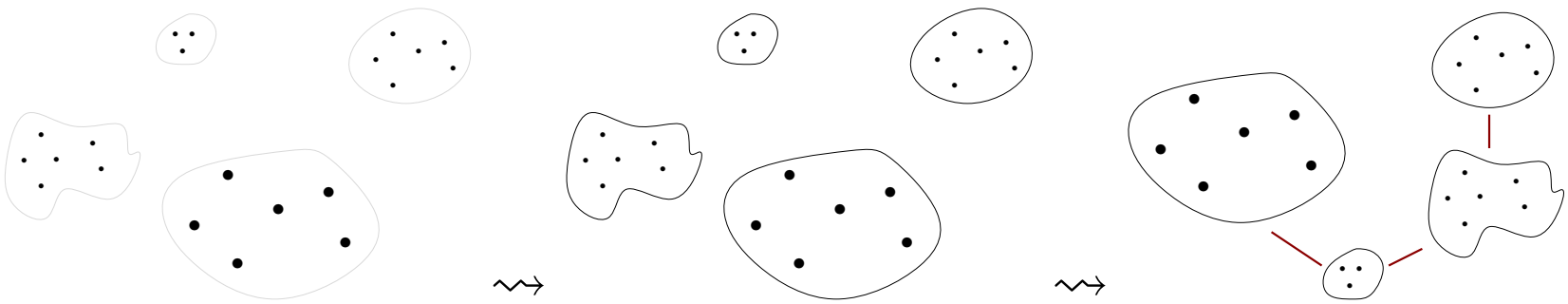
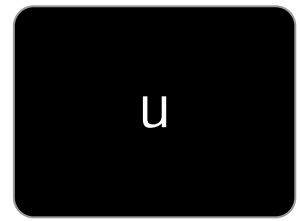


Contextual Equivalence and Preorder



When are two λ-terms equivalent? [Mor68]

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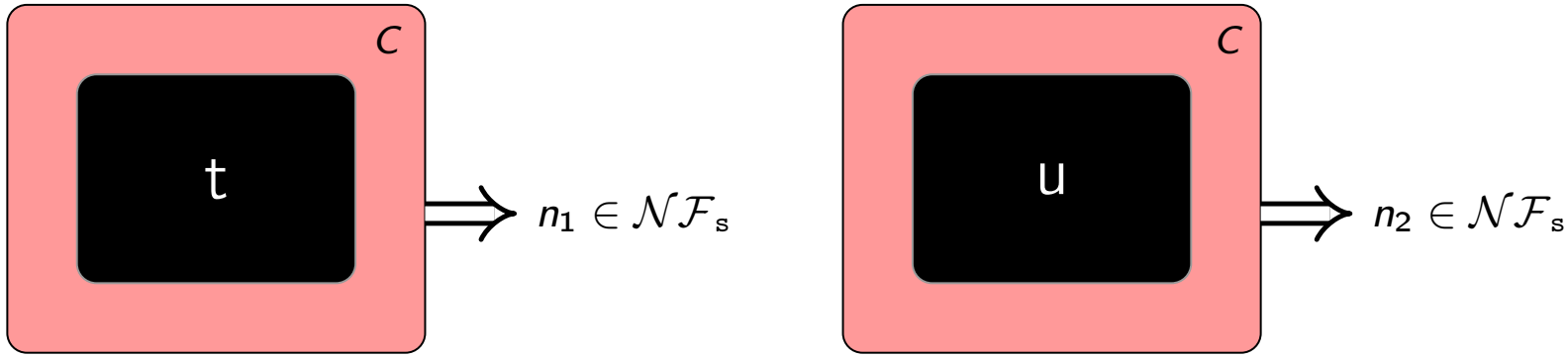

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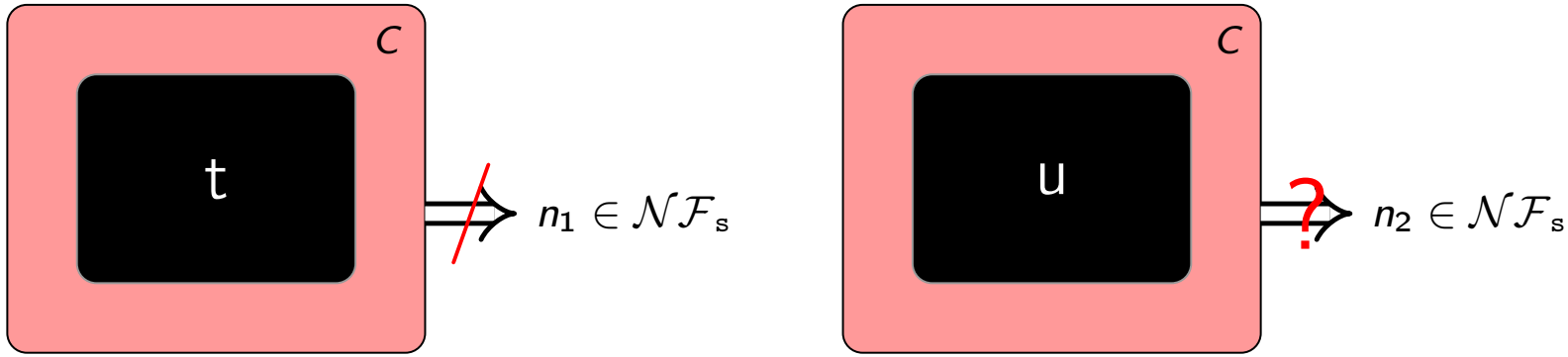

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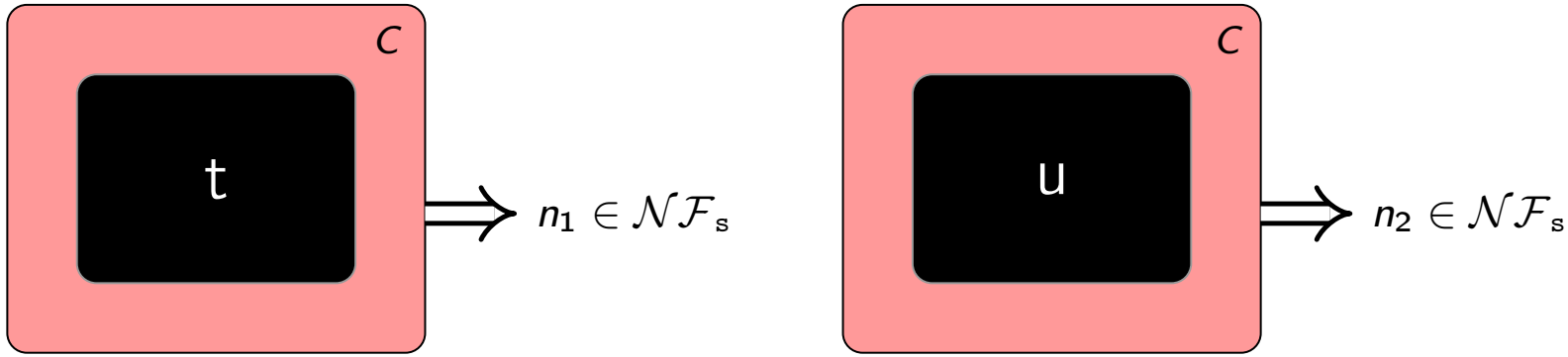

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Why Contextual Equivalence?

The contextual preorder is a natural extensional principle, but that can be quite hard to use.

« The contextual preorder is the largest sensible inequational theory to study. »

Maximal Theory:

A theory $\leq_{\mathcal{T}}$ is maximal if for any theory $\leq_{\mathcal{T}'}$ we have that if $\leq_{\mathcal{T}} \subsetneq \leq_{\mathcal{T}'}$ then $\leq_{\mathcal{T}'}$ is inconsistent.

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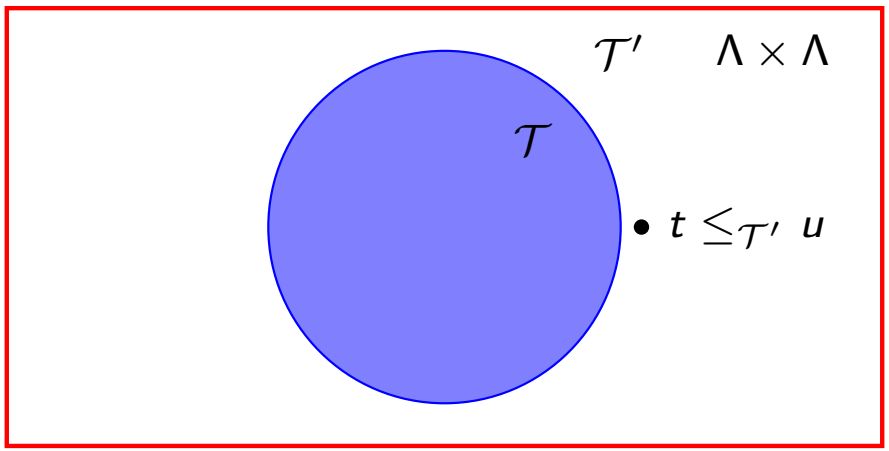

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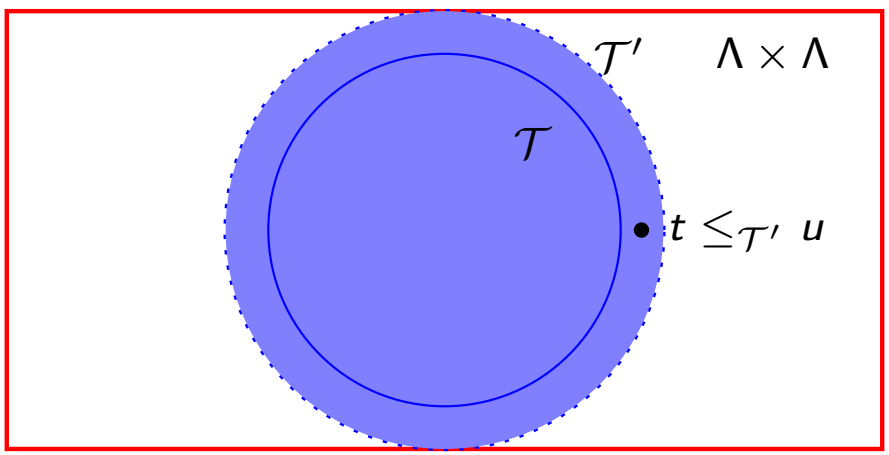

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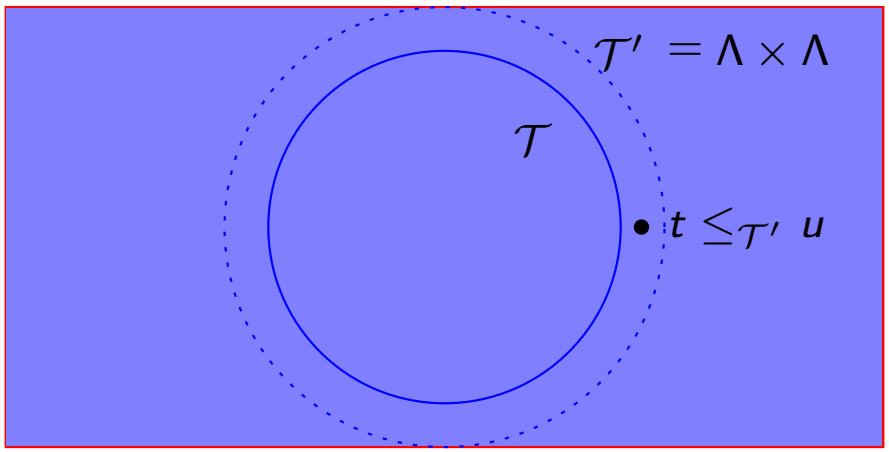

Sufficient Conditions for Maximality

Theorem (**Maximality of Contextual Preorders**)

For s *a* ***dissecting*** *strategy which enjoys light genericity, the contextual preorder* $\sqsubseteq_{=\mathbf{s}}^{\mathrm{octx}}$ *is maximal.*

Factorizing: Proofs $\rightarrow$ 1 parametric proof via **dissecting** strategies:

for every $t \Downarrow_{\mathsf{s}}$ and $u \not\Downarrow_{\mathsf{s}}$, for every s there exists C such that $C\langle t\rangle =_{\mathsf{s}} s$ and $C\langle u\rangle \not\Downarrow_{\mathsf{s}}$.

Simplifying: CbN proof relied on Böhm's theorem $\rightarrow$ proof relying on **light genericity**:

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Light Genericity in Call-by-Name

Light Genericity in Call-by-Name is stated using head reduction.

CbN Light Genericity: head-diverging terms are minimum for the head open contextual preorder.

Which unfolds to:

CbN Light Genericity: let u be head-diverging and C such that $C\langle u\rangle$ is head-normalizing then $C\langle t\rangle$ is head-normalizing for all $t \in \Lambda$.

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Proofs of Call-by-Name (Light) Genericity

- Original Topological proof by Barendregt
- Operational Proof by Takahashi
- Taylor Expansion proof of Barbarossa and Manzonetto
- Approximants Calculus by Arrial, Guerrieri and Kesner
- Proofs using Semantic Models
- Many Others

Light Genericity in Call-by-Name

Takahashi proves heavy genericity with a very short proof [Tak94] and gives as a corollary light genericity.

Key idea: Reason with substitutions instead of contexts!

In the paper, we adapt **Takahashi's technique** to give a direct proof of light genericity.

Light genericity as substitution: let u be s-diverging and t such that $t\{x \leftarrow u\}$ is s-normalizing then $t\{x \leftarrow s\}$ is s-normalizing for all $s \in \Lambda$.

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Formalizing λ

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$$\frac{}{(\lambda x.t)u \to_{\mathtt{h}} t\{x \leftarrow u\}} \qquad \frac{t \to_{\mathtt{h}} u}{ts \to_{\mathtt{h}} us} \qquad \frac{t \to_{\mathtt{h}} u}{\lambda x.t \to_{\mathtt{h}} \lambda x.u}$$

Formalizing λ-theories

Definition (Inequational theory)

An inequational theory $\leq_{\mathcal{T}}$ is a pre-order on terms such that:

- *Stable by contextual closure*:
  $t \leq_{\mathcal{T}} u \implies \forall C,\ C\langle t\rangle \leq_{\mathcal{T}} C\langle u\rangle$;
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Head (open) contextual preorders is an inequational theory.

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Rewriting results: Normalization

Theorem (Head Normalization)

Let t be a term. If $t \to_\beta^* u$ and u does not $\to_{\mathtt{h}}$-reduce then $\to_{\mathtt{h}}$ terminates on t.

Proven by factorization/standardization.
(We also formalize confluence but come one everyone has done it once in their life!)

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Solvability Characterization

The following are equivalent, given a term t:

- *Solvability*: there exists a head context H such that $H\langle t\rangle \to_\beta^* \mathtt{I} := \lambda x.x$;
- *Head Normalization*: t is head normalizing.

Proven via the head normalization theorem.

Formalizing Contexts

A context is a term with a hole? Not really...

For $C := \lambda x.\langle\cdot\rangle$, we have that $C\langle x\rangle = \lambda x.x$ and $C\langle y\rangle = \lambda x.y$

A context is a term with a hole *annotated by the possible captured variables*...

$$\lambda x.\Box_x$$

A context is a function from `tm` to `tm`, coded in Abella as a predicate:

`ctx T CT` holds iff there exists a context C such that $C\langle \texttt{T}\rangle = \texttt{CT}$.

Applying a context to two different terms:

`ctxs T CT U CU` holds iff there exists a context C such that $C\langle \texttt{T}\rangle = \texttt{CT}$ and $C\langle \texttt{U}\rangle = \texttt{CU}$.

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Takahashi's Trick in CbN

$$C\langle u\rangle \leftrightarrow t_C\{x{\leftarrow}u_C\}$$
$$C\langle s\rangle \leftrightarrow t_C\{x{\leftarrow}s_C\}$$

Trick:
Let $\mathtt{fv}(u) \cup \mathtt{fv}(s) = \{x_1, \ldots, x_k\}$, and y a fresh variable.

- $u_C := \lambda x_1 \ldots \lambda x_k.u$ and $s_C := \lambda x_1 \ldots \lambda x_k.s$ are closed terms.
- Consider $t_C := C\langle y x_1 \ldots x_k\rangle$, and note that:

$$\begin{array}{rcc} t_C\{y{\leftarrow}u_C\} & = & C\langle u_C x_1 \ldots x_k\rangle \\ & = & C\langle (\lambda x_1 \ldots \lambda x_k.u) x_1 \ldots x_k\rangle \\ & \to_\beta^k & C\langle u\rangle \end{array}$$

- u is h-diverging implies that u_C is also h-diverging.
- $C\langle u\rangle$ is h-normalizing if and only if $t\{y{\leftarrow}u_C\}$ is. (also true for s and s_C)

  by the Head Normalization Theorem (and confluence, etc.)

Formalizing Takahashi's Trick

Lemma (Disentangling)

Let C be a context. Then there exist t_C and a variable $x \notin \mathtt{fv}(C)$ such that for all terms u there exists u_C such that $t_C\{x \leftarrow u_C\} \to_\beta^* C\langle u \rangle$. Moreover, if u is head divergent then u_C is head divergent.

Some technicalities in Abella...

Definition (Substitution preorder)

The *substitution preorder* $t \precsim_{\mathcal{S}}^{\mathtt{h}} u$ holds when $s\{x \leftarrow \lambda y_1 \ldots \lambda y_n.t\}$ $\to_{\mathtt{h}}$-terminating implies that $s\{x \leftarrow \lambda y_1 \ldots \lambda y_n.u\}$ is $\to_{\mathtt{h}}$-terminating for all terms s, variables x, and lists of variables $y_1, \ldots, y_n$ with $n \geq 0$.

Back to Light Genericity

We can show (in Abella) light and heavy genericity using this disentangling lemma.

We also show that the head contextual preorder includes β-conversion.

Relies on head normalization, confluence, etc.

Which lead us to:

- The consistency of collapsing $\mathtt{h}$-diverging terms, and;
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The proof of maximality always starts by:

If $\mathcal{T} \vdash t \leq u$ and $t \not\preceq^{\mathrm{octx}}_{\mathtt{h}} u$
Then $\underline{\exists C}$ such that

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Why I think everyone[1] should start using Abella

and why I might stop using it

- Easy to get started with, only few tactics
- Pen-and-paper proofs translate easily

- Expressive power is limited (I would like to quantify over relations)
- Missing some "quality of life" features

  Reasoning with integers can already be painful...

[1] in the comprehensive set of people interested in functional languages with binders

Conclusions

- Towards a Formal Companion of Barendregt's book
- Easy proofs that rely mostly on rewriting/operational results
- Formalization faithful to pen-and-paper proofs

Future work:

- Can be adapted to CbV but results are not formalized (also a lot of the results can be factorized)
- Constructive Contextual (In)Equivalence?
- Other results on program equivalence to be made formal (mechanizing Böhm trees and Böhm's theorem?)

Thank you!

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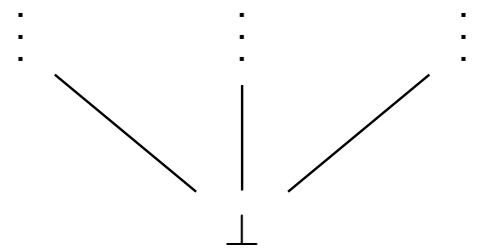
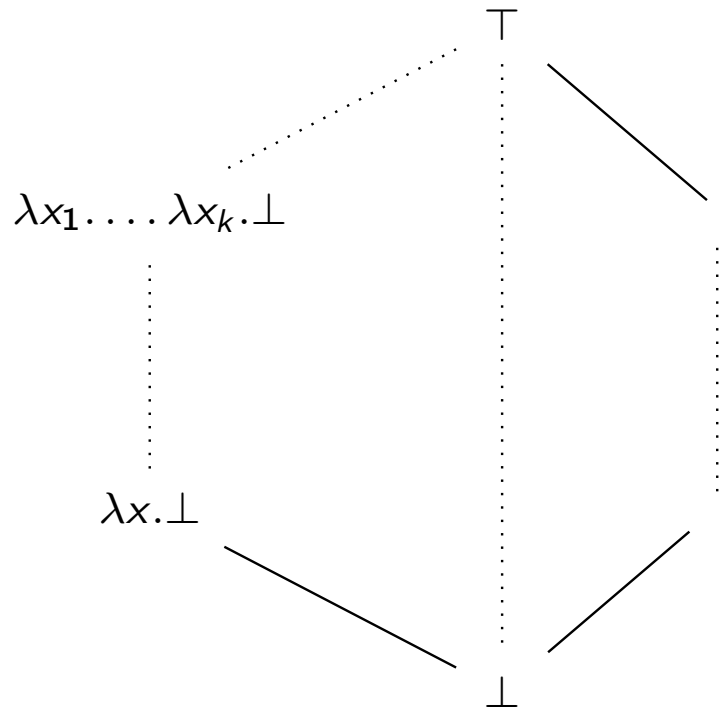


CbN



CbV

Contextual preorder for lambda terms

$\bot :=$ equivalence class of Ω

Formalizing λ

$$\text{Terms} \quad t, u \quad := \quad x \mid \lambda x.t \mid tu$$

λ-terms and the predicate for inducting on them in Abella:

```
Kind  tm   type.

Type  abs  (tm -> tm) -> tm.
Type  app  tm -> tm -> tm.


Define is_tm : tm -> prop by
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```
Define beta : tm -> tm -> prop by
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Formalizing λ and β

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Formalizing λ-theories

A λ-theory is stable by contexts...

Contextual equivalence...

We need to formalize contexts

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Formalizing Contexts

A context is a term with a hole? Not really...

Set $C := \lambda x.\langle\cdot\rangle$, then $C\langle y\rangle = \lambda x.y$ and $C\langle x\rangle = \lambda x.x = \mathtt{I}$.

`ctx T CT` holds iff there exists a context C such that $C\langle \mathtt{T}\rangle = \mathtt{CT}$.

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Define ctx : tm -> tm -> prop by
  ctx T T;
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  nabla x, ctx (T x) (abs CT) :=
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How to apply a context to two different terms?

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`ctxs T CT U CU` holds
iff there exists a context C such that $C\langle \texttt{T} \rangle = \texttt{CT}$ and $C\langle \texttt{U} \rangle = \texttt{CU}$.

```
Define ctxs : tm -> tm -> tm -> tm -> prop by
  ctxs T T U U;
  ctxs T (app A B) U (app C D) :=
    (ctxs T A U C /\ B = D /\ tm D)
    \/ (ctxs T B U D /\ A = C /\ tm C);
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```

Formalizing Contextual Preorder

```
Define ctx_preord : tm -> tm -> prop by
  ctx_preord P Q := forall CP CQ,
    tm P -> tm Q ->
    ctxs P CP Q CQ -> head_terminating CP ->
    head_terminating CQ.
```

- `ctx_preord` is stable by contexts.
- `ctx_preord` is invariant under computation.
- `ctx_preord` has `h`-diverging terms as minimums.

Light Genericity

Light Genericity: head-diverging terms are minimum for the head open contextual preorder.

Unfolded statement:

Light Genericity: let u be head-diverging and C such that $C\langle u\rangle$ is head-normalizing then $C\langle t\rangle$ is head-normalizing for all $t \in \Lambda$.

Main difficulty: reasoning with contexts and reduction.

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Direct proof of Light Genericity

Takahashi proves Barendregt's heavy genericity with a very short proof [Tak94] and gives as a corollary light genericity.

Key idea/trick: Reason with substitutions instead of contexts!

Light genericity as substitution: let u be $\mathtt{h}$-diverging and t such that $t\{x \leftarrow u\}$ is $\mathtt{h}$-normalizing then $t\{x \leftarrow s\}$ is $\mathtt{h}$-normalizing for all $s \in \Lambda$.

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Takahashi's Trick in CbN

$$C\langle u\rangle \leftrightarrow t_C\{x{\leftarrow}u_C\}$$
$$C\langle s\rangle \leftrightarrow t_C\{x{\leftarrow}s_C\}$$

Trick:
Let $\mathtt{fv}(u) \cup \mathtt{fv}(s) = \{x_1, \ldots, x_k\}$, and y a fresh variable.

- $u_C := \lambda x_1 \ldots \lambda x_k.u$ and $s_C := \lambda x_1 \ldots \lambda x_k.s$ are closed terms.
- Consider $t_C := C\langle y x_1 \ldots x_k\rangle$, and note that:

$$\begin{array}{rcc} t_C\{y{\leftarrow}u_C\} & = & C\langle u_C x_1 \ldots x_k\rangle \\ & = & C\langle (\lambda x_1 \ldots \lambda x_k.u) x_1 \ldots x_k\rangle \\ & \to_\beta^k & C\langle u\rangle \end{array}$$

- u is h-diverging implies that u_C is also h-diverging.
- $C\langle u\rangle$ is h-normalizing if and only if $t\{y{\leftarrow}u_C\}$ is. (also true for s and s_C)

  by the Head Normalization Theorem (and confluence, etc.)

Conclusions

- A small subset of Barendregt's book formalized (many rewriting theorems hidden in this presentation)
- Easy proofs that rely mostly on rewriting/operational results
- Faithful formalization of the pen-and-paper proofs

Future work:

- Constructive Contextual (In)Equivalence?
- Many results adapt to the theory of the Call-by-Value calculus (haven't formalized these)
- Other results on program equivalence to be made formal (mechanizing Böhm trees and Böhm's theorem?
  $\Rightarrow$ intensional presentation of contextual equivalence)

Thank you!

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Thank you!


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